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Efficiency Effects on Coalition Formation in Contests

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Abstract

This paper studies the problem of endogenous coalition formation in contests: how players organize themselves in groups when faced with the common objective of securing a prize by exerting costly effort. The model presented adopts an axiomatic approach by assuming certain properties for the winning probability that imply efficiency gains from cooperation in contest settings. Efficiency gains are said to be generated if any coalition experiences increasing marginal returns with aggregate effort until a threshold. These properties identify a wide class of generalised Tullock contest success functions.

We analyse a sequential coalition formation game for an arbitrary number of symmetric players and exogenous effort. If coalitions generate sufficient efficiency gains, then any equilibrium always consists of two or more coalitions where at least two coalitions are of unequal size. This result extends to endogenous efforts if the cost functions are sufficiently convex.

Keywords: Noncooperative Games, Coalition Formation, Contest Success Function

JEL Classification: C72, D74

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1 Introduction

Most economic, social and political activities involve the formation of competing groups (e.g., political parties, institutions, mergers or partial cartels) in self-interest. This paper studies the equilibrium outcomes in such settings where individual players endogenously organize groups with the mutual target of securing a prize. The literature on endogenous coalition formation in contests concludes that if players experience the incentive to form coalitions, then the equilibrium comprises a maximum of two coalitions: a majority and a minority group. In a three-player model, Skaperdas (1998) and Tan and Wang (1997) find that if cooperation satisfies the increasing returns to scale property, then a two-player alliance competes against the remaining individual player. Tan and Wang (2010) extend this idea to show that the two-group outcome applies to n-players if the cooperation displays the increasing returns property. In absence of this property, full cooperation among players - the grand coalition - emerges as the outcome[1].

However, this literature fails to explain the formation of three or more competing coalitions. The lack of a theory to explain this phenomenon is attributed to the restrictive nature of contest models employed in the literature. Either these models apply variants of the Tullock contest success function, that are highly stylized, or they limit their analysis to a few players - three, four, or five - that restrict possible combinations of the coalition structures. Although these restrictions simplify the analysis, it leads to an incomplete conclusion: if the incentive to cooperate exists, then either a bipartite coalition structure or the grand coalition emerges.

This paper adopts an axiomatic approach by assuming certain properties satisfied by the winning probability function. These properties identify a wide class of generalized Tullock contest success functions where cooperation generates efficiency gains: the impact of the effort exerted by members of a coalition on its winning probability is greater than its aggregate (Proposition 1). We analyze a sequential coalition formation game proposed by Bloch (1996) for n symmetric players and exogenous effort under the restrictions imposed by these properties.

We show that any equilibrium always consists of two or more coalitions where the size of at least one coalition differs from the rest in the equilibrium coalition structure (Theorem 1). An important implication of this result is that the possibility of forming three of more coalitions exists (Corollary 1.1). This result carries over to endogenous efforts whenever the cost of exerting efforts is sufficiently convex (Proposition 3). Thus, there are three key contributions of this paper: we show that the possibility of forming three or more coalitions exists; we identify and formalise the

[1]Bloch, Sanchez-Pages and Soubeyran (2006), Sanchez-Pages (2007), and Bloch (2012) show that full cooperation (or grand coalition) by players is often the outcome when the incentive to form groups exists.

economic principles that lead to this result; we show that these economic principles are satisfied by a wide class of generalized Tullock contest success functions.

The model presented in this paper follows a two-stage process. First, players endogenously form coalitions in stage one. For that purpose, we specify a coalition formation mechanism: a set of rules or a procedure through which players carry out negotiations to form coalitions. We adopt the mechanism proposed by Bloch (1996) and Ray and Vohra (1997) where players sequentially form coalitions through a proposer-responder protocol. In this exclusive membership game, a randomly chosen player proposes a coalition. If any player among the proposed members rejects the offer, then a new proposer is chosen randomly. However, once players unanimously agree on the coalition, deviations are not permitted at any stage of the game. That is, the coalition contract is binding. After this negotiation process ends and a coalition forms, this procedure is repeated with the remaining players. This game ends when all negotiations are completed. The resulting outcome is a coalition structure and its stability is given by the *stationary perfect equilibrium*: a refinement of the sub-game perfect equilibrium that requires the strategy at each stage of the game to be independent of the history of actions. In stage two, the coalitions engage in a contest where a coalition's winning probability depends on the aggregate effort of its members, and that of rival coalitions.

Within this framework, we restrict the partition function - interpreted as the winning probability - through three assumptions. Assumption 1 states the axioms of probability, monotonicity, and anonymity based on Skaperdas (1996); that was later generalized by Münster (2009) to group contests. Assumption 2 relates to the notion of eventually diminishing marginal benefit. That is, the marginal benefit from effort increases up to a threshold, and eventually diminishes beyond that threshold. Although this assumption is a common feature of several contest settings, it has neither been formalised in the contest not coalition formation literature.[2] This is another contribution of this paper. Assumption 3 states that the efforts behave as strategic substitutes until a threshold, and then as strategic complements. Assumptions 1 and 3 are ubiquitous: any Generalised Tullock contest success function satisfies both assumptions. However, Assumption 2 is satisfied by the class of Generalised Tullock contest success functions where cooperation generates efficiency gains. That is, situations where the impact of a coalition's effort on its winning probability is larger than its aggregate.[3]

[2]In *R&D contests* where Loury (1979) and Lee and Wilde (1980) assume that the winning probability displays diminishing marginal benefit; *Wars of attrition* where Riley(1979) and Nalebuff and Riley (1985) use a similar assumption. Dixit (1987) assumes strict concavity in a general contest setting.

[3] This implies that the impact factor - a standard terminology in the contest literature - is greater than 1. Refer

We analyse this model for exogenous efforts. Our main result establishes that if a standard contest success function (Assumption 1) displays some degree of the increasing returns property (Assumption 2), in addition to eventual strategic complementarity (Assumption 3), then the equilibrium is an intermediate coalition structure that is non-symmetric: at least one coalition's size differs from the rest. Thus, neither the extreme of the grand coalition forms nor the other extreme of no coalitions occurs.

The intuition for this result is as follows. Under the assumption of exogenous effort, any coalition can increase its winning probability only by admitting members. We show that the marginal benefit of adding members to a coalition - through pair-wise transfers- first increases until a threshold, and then begins to diminish beyond that. Hence, this property introduces the incentive to cooperate due to the inceptive efficiency gains from adding members. If the grand coalition forms, there is no value addition from efficiency gains as all players divide the prize equally. Therefore, it is profitable to form a majority coalition - to benefit from efficiency gains - and engage in a contest compared to forming the grand coalition.

Although there always exists a bipartite structure that dominates the grand coalition, it is not necessarily the equilibrium. The equilibrium may consist of three or more coalitions depending on the size of the threshold coalition (mentioned above) and the nature of the increasing/decreasing returns. The number of coalitions formed at equilibrium is likely to be high if: (1) the threshold coalition size is low; (2) the returns from adding members diminishes rapidly beyond that threshold.

Lastly, this result extends to the situations of endogenous effort choice and sufficiently convex cost functions. This continuity of results is observed because the exogenous effort model is a special case of convex cost functions. In the exogenous effort model, the cost of increased effort from zero to one is negligible, and beyond one is infinite. Hence, sufficiently convex cost functions produce an effect similar to exogenous effort models.

■ **Literature Review**: This paper belongs to the literature on coalition formation with binding agreements in contests. That is, once a coalition wins the contest, the members credibly implement a sharing rule for the prize that prevents any further conflict among the players. We adopt a sequential mechanism of coalition formation based on Bloch (1996) where the coalitional worth depends on the coalition structure and is distributed among the members according to a credible predetermined rule. Ray and Vohra (1999) follow a similar coalition formation mechanism, but where coalitional worth is distributed among the members through an endogenous negotiation process. Both papers

to Proposition 1 (Section 4) for more details.

explicitly model the coalition formation process as a non-cooperative sequential bargaining process and characterize stationary sub-game perfect equilibria in the spirit of Rubinstein's (1982) bargaining game. The solution concept implemented in our paper applies the mechanism developed by these two papers.

A related paper by Bloch et al. (2006) implements the above coalition formation mechanism in a symmetric rent-seeking model where the players exert costly effort to win a prize. The unique sub-game perfect equilibrium is the grand coalition in their paper. However, this finding depends on the fact that adding players to a coalition fails to generate efficiency gains. Hence, if a sub-group deviates from the grand coalition their expected share of the prize remains unchanged, but the deviating group incurs a cost due to competition. Therefore, the grand coalition is stable. However, if coalitions generate sufficient efficiency gains - due to the increasing returns property stated by Assumption 2 - then a deviation from the grand coalition may yield a higher expected share of the prize that offsets the cost incurred. In Section 4, we show that such deviations occur in a model endogenous effort; that extends our exogenous effort model.

Another related paper by Tan and Wang (2010) studies endogenous coalition formation with heterogeneous players and assumes a specific functional form for contest success function. They show that if agreements are binding, then the equilibrium coalition structure for n players is bipartite: only two coalitions are competing against each other in the *initial* contest. On similar lines, Noh (2002), Garfinkel (2004), Changxia, Konrad, and Morath (2011), and Sanchez-Pages (2007a,b) study coalition formation in a symmetric model of contests in which coalitional payoffs are exogenously specified. Common features of these models include a specific contest success function to describe the nature of the conflict. They conclude that either a bipartite coalition structure or the grand coalition emerges an equilibrium outcome. All papers on this topic use a specific variant of the Tullock success function for modeling the winning probability, while our paper generalizes the setting by adopting the axiomatic approach that imposes certain assumptions on the winning probability. This generalization yields an insight that is not present in literature: the formation of three or more coalitions exists.

Separate literature applies the "continuing conflict" approach to study coalition formation in the context of conflict where if a coalition wins, the allies within the coalition are unable to commit to certain sharing rules and hence in turn compete among themselves to decide who should have the prize, leading to further conflicts until one individual winner is left. Skaperdas (1998) was the first to consider this problem with three heterogeneous players. His model assumes exogenous effort terms as *strategic endowment*: a number representing the level of effort a player can exert in any contest she was to enter. Two players may form a coalition and pool in their effort to compete against the third player. If the coalition wins, then the two players engage in a second contest among themselves. The main result in his paper is that a stable alliance forms between two players against the third player if and only if the CSF exhibits superadditive or increasing-returns property concerning the strategic endowments of at least two players. When the CSF is always subadditive, all three players stand alone in a three-way contest. Tan and Wang (2010) also study this problem of endogenous coalition formation with heterogeneous players and exogenous effort through *continuing* conflict. They show that the equilibrium coalition structure for n players is bipartite: only two coalitions are competing against each other in the *initial* contest. However, unlike the binding contracts approach, they are unable to characterize the entire equilibrium of sub-coalitions in subsequent contests which is a defining aspect of continuing conflict. Esteban and Sakovics (2003) consider a three-player model, but with endogenous efforts. They show that absent synergy (efficiency gains), no coalition formation occurs because individual participation exceeds that from continuing conflict in case of forming a coalition. Thus, although this is different from our paper, this literature also concludes that if the incentive to form coalitions exists, then either a bipartite structure forms or no coalitions form.

The paper is structured as follows: we describe the model in *Section 2* that involves stating the underlying assumptions and the coalition formation game. In *Section 3* we analyze the case of exogenous effort. *Section 4* presents an extension for endogenous efforts. The conclusions are stated in *Section 5*.

2 The Model

A set of identical and risk neutral players $N = \{1, 2..., n\}$, where $n \geq 3$, compete to win a prize of value normalised to unity. Players may endogenously form mutually exclusive coalitions that result in a coalition structure - partition of N - denoted by $\Pi = \{C_1, C_2..., C_K\}$. Let $\mathbb{P}$ represent the set of all coalition structures, and 2^N the power set of N.

Once the coalition structure forms, each player $i \in N$ exerts an effort $y_i \geq 0$ at cost $c(y_i)$. Let $\mathbf{Y}$ denote the vector of efforts exerted by all players. Suppose $i \in C_k \in \Pi$, then the vector of individual efforts by remaining members of that coalition is $\mathbf{y}_{-i}^{C_k}$. The aggregate effort expended by any coalition is $Y_{C_k} = \sum_{j \in C_k} y_j$, and the vector of aggregate efforts of all remaining coalitions is $\mathbf{Y}_{-C_k}$.

The probability that coalition C_k wins - termed its winning probability - is given by the partition function $p : 2^N \times \mathbb{P} \times \mathbb{R}_+^n \to [0, 1]$ where $p(C_k, \Pi, \mathbf{Y}) \geq 0$ only if $C_k \in \Pi$. The partition function is continuous and differentiable with respect to the effort of any player $i \in N$.

If C_k wins the contest, then it implements an effort dependent sharing rule $s : \mathbb{R}_+^{|C_k|} \to [0,1]$ where $s(y_i, \mathbf{y}_{-i}^{C_k})$ is player i's share of the prize. We assume the sharing rule $s(.)$ is common knowledge, exogenous and binding. For any two players $i, j \in C_k$, if $y_i \leq y_j$, then player j (who exerts more effort) receives at least as much as player i. That is, $s(y_i, \mathbf{y}_{-i}^{C_k}) \leq s(y_j, \mathbf{y}_{-j}^{C_k})$. The sum of shares of all members of the winning coalition adds up to the total prize: $\sum_{i \in C_k} s(y_i, \mathbf{y}_{-i}^{C_k}) = 1$; implying no distributional efficiency loss.

Thus, for a given coalition $C_k \in \Pi$, the payoff to any player $i \in C_k$ is the difference between the expected value of his share of the prize and the cost of exerting effort.

$$u_i(C_k, \Pi, \mathbf{Y}) = s(y_i, \mathbf{y}_{-i}^{C_k}) p(C_k, \Pi, \mathbf{Y}) - c(y_i) \tag{1}$$

The remaining section proceeds as follows: First, we state the assumptions imposed on the partition function $p(.)$ in subsection 2.1. We then state the coalition formation game in subsection 2.2.

2.1 Central Assumptions

The first assumption imposes primary axioms fundamental to group contests based on Münster (2009); which were derived from axioms for an individual contest by Skaperdas (1996).

Assumption 1. *For any coalition structure* $\Pi \in \mathbb{P}$ *and effort vector* $\mathbf{Y} \in \mathbb{R}_+^n$

(A) $\sum_{C_k \in \Pi} p(C_k, \Pi, \mathbf{Y}) = 1$ *where* $p(C_k, \Pi, \mathbf{Y}) > 0$ *if and only if* $y_i > 0$ *for at least some* $i \in C_k$.

(B) $p(C_k, \Pi, \mathbf{Y})$ *strictly increases (resp. decreases) with an increase (resp. decrease) in the effort of any member* $i \in C_k$ *(resp.* $j \notin C_k$*) where* $i, j \in N$.

(C) *For any* $i \in C_k$ *and* $j \in C_\ell$, *if* $y_i = y_j$ *then* $p(C_k, \Pi, \mathbf{Y}) = p(C'_k, \Pi', \mathbf{Y})$ *where* $C'_k = (C_k \setminus \{i\}) \cup \{j\}$, $C'_\ell = (C_\ell \setminus \{j\}) \cup \{i\}$, *and* $\Pi' = (\Pi \setminus \{C_k, C_\ell\}) \cup \{C'_k, C'_\ell\}$.

Assumption (*1A*) states that the partition function generates a probability distribution for any effort vector over any coalition structure. Assumption (*1B*) states that if any player increases effort, the winning probability of his coalition increases, but decreases the winning probability of all remaining coalitions. Assumption (*1C*) refers to between-group anonymity, i.e. the identities of the groups do not matter. As players are identical, within-group anonymity is implied. Hence, a coalition's winning probability is solely determined by the level of effort exerted by its constituent players; their identities are irrelevant.

The next two assumptions impose restrictions beyond the standard axiomatization of contest success functions in literature. They are motivated by the standard generalised Tullock contest success functions. Specifically, the Assumption 2 applies to settings where cooperation generates efficiency gains, while Assumption 3 applies to any generalised Tullock contest success functions.[4]

Assumption 2. *(Convex-Concavity)*
For any $C_k \in \Pi$, there exists a threshold $z_1(p(.), \Pi, \mathbf{Y}_{-C_k}) > 0$ such that $p(C_k, \Pi, \mathbf{Y})$ is strictly concave (resp. convex) with respect to coalition C_k's effort for all $Y_{C_k} >$ (resp. $<$) $z_1(p(.), \Pi, \mathbf{Y}_{-C_k})$. The threshold $z_1(p(.), \Pi, \mathbf{Y}_{-C_k})$ increases with an increase in the aggregate efforts Y_{C_j} of any rival coalition $C_j \neq C_k$.

This implies that there exists a point of inflection at which the curvature of the winning probability function changes its sign. Hence, coalitions experience increasing marginal benefit by increasing efforts if their aggregate effort is below $z_1(.)$. Otherwise, the coalition experiences diminishing marginal benefit by increasing efforts. This threshold, $z_1(p(.), \Pi, \mathbf{Y}_{-C_k})$, is specific to coalition C_k and depends on the partition function $p(.)$, the coalition structure Π, and the vector of aggregate effort by the remaining coalitions $\mathbf{Y}_{-C_k}$. The effort region with increasing return, $[0, z_1(p(.), \Pi, \mathbf{Y}_{-C_k})]$, expands with higher effort of rivals.

The next assumption states the condition when the winning probability displays complementary and substitutability with aggregate efforts.

Assumption 3. *(Eventual Supermodularity)*
For any $C_k \in \Pi$, there exists a threshold $z_2(p(.), \Pi, \mathbf{Y}_{-C_k}) \geq z_1(p(.), \Pi, \mathbf{Y}_{-C_k}) > 0$ such that $p(C_k, \Pi, \mathbf{Y})$ is supermodular (resp. submodular) with respect to C_k's effort and the effort of any rival coalition for all $Y_{C_k} >$ (resp. $<$) $z_2(p(.), \Pi, \mathbf{Y}_{-C_k})$. The threshold $z_2(p(.), \Pi, \mathbf{Y}_{-C_k})$ increases with an increase in the aggregate efforts Y_{C_j} of any rival coalition $C_j \neq C_k$.

This implies that there exists a threshold point at which the winning probability function changes its nature from being submodular to supermodular. As with the previous threshold, $z_2(p(.), \Pi, \mathbf{Y}_{-C_k})$ is also specific to coalition C_k and depends on the partition function, the coalition structure, and the aggregate effort by the remaining coalitions.

The supermodularity of the winning probability implies that an increase in one coalition's efforts increases the marginal payoff of action for all its rival coalitions. That is, if coalition C_k chooses a higher effort level, all other coalitions have an incentive to raise their effort levels too. Similarly,

[4] Generation of efficiency gains means that the impact of the effort exerted by members of a coalition is greater than its aggregate effort. Refer to Proposition 1 for details.

submodularity of the winning probability implies that an increase in one coalition's efforts decreases the marginal payoff of action for all its rival coalitions.

2.2 Coalition formation game

Until now, we have discussed the structure assumptions 1 - 3 impose on the contest success function, and hence on the payoff function. In this section, we consider a coalition formation mechanism based on Bloch(1996). Our objective is to study the nature of the equilibrium coalition structure that arises from the structure imposed on the payoff functions in this contest game.

We now begin describing the mechanism that is based on Bloch (1996). Players sequentially form coalitions through a proposer-responder protocol: given that some coalitions have already formed, players make proposals to coalitions and respond to proposals made to coalitions to which they belong to. The proposer must include herself in the coalition. Once a coalition is proposed, its members respond, according to a given order, by either accepting or rejecting the proposal. The responders either accept the proposal or reject it. The coalition forms if the responders unanimously accept the proposal. Each remaining set of active players is assigned a random probability distribution over proposers. The game then continues between the remaining active players.

If a responder rejects, it creates bargaining friction: the payoff is discounted by δ for all players. The rejector may choose to leave, effectively forming a one-person coalition. If not, then the rejector is chosen as a proposer in the next round with probability ρ and some other member with $1-\rho$.

We formalize this mechanism based on Ray and Vohra (1997). Let $\pi \subset \Pi$ be a coalition sub-structure. Define $K(\pi) = \bigcup_{C\in\pi} C$ and $K(\emptyset) = \emptyset$ where $\emptyset$ denotes the null sub-structure. Let $\mathbb{F}$ denote the family of all sub-structures. Given that some coalitions have formed the sub-structure π, the set of players yet to form coalitions, $N \setminus K(\pi)$, are termed as *active.*

Define a function $f : \mathbb{F} \to 2^N$ such that $f(\pi)$ assigns a coalition, that is not yet in π, to a given sub-structure π. Define

$$c^f(\pi) = \pi \cup \{f(\pi), f\big(\pi \cup f(\pi)\big), ...\}$$

The vector $c^f(\pi)$ is interpreted as the coalition structure that forms, given π is already formed. For any π, the equilibrium action of a proposer is $\sigma_i^P(\pi, N) = f(\pi)$ to be the largest coalition $C \subseteq N \setminus K(\pi)$ that maximises the average winning probability $\frac{p\left(f(\pi),c\left(\pi\cup f(\pi)\right)\right)}{f(\pi)}$.[5] This tie-breaking rule implies that at every step, players form the largest coalition which maximises their average

[5]This tie-breaking rule incentivizes players towards forming the grand coalition. However, this assumption is not important for our main results.

winning probability, calculating the actions of the players that remain after they form a coalition.

If the player is a responder and the coalition $f(\pi)$ is proposed, the responder's action is $\sigma_i^R(\pi, N, f(\pi)) = \{Yes\}$ or $\{No\}$. This is a stationary strategy as the information set for the proposer does not depend on the history of proposals offered and rejections made. It depends only on the current state $\{\pi, N \setminus K(\pi)\}$; π is the coalition sub-structure formed and $N \setminus K(\pi)$ is the set of active players.

A stationary perfect equilibrium is a profile of stationary strategies $\{\sigma\}_{i \in N}$ such that there is no round at which a player benefits by deviating from her prescribed strategy. The equilibrium coalition structure is given by $\Pi^* = c(\emptyset)$.

3 Exogenous effort model

So far, we have described a two-stage process: in stage one players form coalitions, and in stage two each player chooses their effort levels. At this level of generality - where the model involves imposing intuitive assumptions - analysing these two stages is non-tractable. Therefore, we introduce an interim step by assuming that every player exerts a constant and symmetric effort level in this section. Specifically, every player exerts a constant effort level normalised to unity such that the effort vector is a unit vector denoted by

$$\bar{\mathbf{Y}} = \{\underbrace{1, 1..., 1}_{n-\text{times}}\}.$$

This effectively eliminates the effort choice game in stage two and allows us to focuses on the coalition formation game played in stage one. The implication of this exogenous effort assumption is the absence of the free-rider effect in our analysis. That is, the size of a coalition does not affect the player's incentive to exert effort. However, by continuity this result extends to situations where the free-rider effect is sufficiently low. In section 4 we identify conditions on the cost function for which the free-rider effect is sufficiently low by using stylised contest success functions.[6]

Given the primitive assumptions on the sharing rule, if coalition C_k wins, then it divides the prize equally among all members in the exogenous effort model.[7] Thus, for any player $i \in C_k$ the

[6] It is worthwhile to mention that this exogenous effort model is interesting in its own right. If the properties derived in Lemmas 1 and 2 are treated as primary assumptions, then they can be used to describe constant sum games where the winning probability is determined by the player type. Tan and Wang (2010) and Skaperdas (1998) treat such exogenous effort models for contest settings. Other applications of this constant sum game with player type is a topic for future research.

[7] The proof for this statement is trivial and we leave it to the interested readers to verify it themselves.

share of the prize is

$$s(y_i, \mathbf{y}_{-i}^{C_k}) = \frac{1}{|C_k|}.$$

Hence, player i's payoff is given by

$$u_i(C_k, \Pi, \bar{\mathbf{Y}}) = \frac{p(C_k, \Pi, \bar{\mathbf{Y}})}{|C_k|} \text{ by applying the normalisation } c(1) = 0. \tag{2}$$

Observe that as the effort level is fixed, members of any coalition can alter their payoff only by admitting new members from rival coalitions. The trade-off of such an action is the following: admitting a new member increases the coalition's winning probability, but decreases each member's share of the prize in case of victory. Thus, coalition C_k adds a new member only if the marginal increase in winning probability offsets the decrease in the individual share of the prize.

The effect of such a transfer on the winning probability is unclear. Admitting a new member from coalition C_j to C_k simultaneously increases C_k's, but decreases C_j's winning probability. Further, such transfers also affect the winning probabilities of the remaining coalitions in Π. Therefore, we need to understand how assumptions 1-3 affect the winning probabilities of all coalitions in Π if any individual, or a sub-group of members, are transferred from one coalition to another.

Lemma 1. *In the exogenous effort model, Assumption 1 implies the following properties*

(P1) $\sum_{C_i \in \Pi} p(C_i, \Pi, \bar{\mathbf{Y}}) = 1$ *for all* $\Pi \in \mathbb{P}$

(P2) If $|C_j| \geq |C_k|$, *then* $p(C_j, \Pi, \bar{\mathbf{Y}}) \geq p(C_k, \Pi, \bar{\mathbf{Y}})$ *for all* $C_j, C_k \in \Pi$ *and* $\Pi \in \mathbb{P}$

(P3) $p(C_j, \Pi, \bar{\mathbf{Y}}) = p(C_k, \Pi, \bar{\mathbf{Y}})$ *if and only if* $|C_j| = |C_k|$ *for all* $C_j, C_k \in \Pi$ *and* $\Pi \in \mathbb{P}$

Proof. See Appendix. □

This lemma states the implication of imposing the exogenous effort assumption. (P1) means that the sum of winning probabilities adds to a constant number (unity); (P2) means that a coalition's winning probability is increasing with its size; (P3) means that only the coalition's size (a number) matters, and not the identity of the players. In this exogenous effort model, properties (P1)-(P3) are a straightforward implication of assumption 1.

In our next result we study the effect assumptions 2 produce in this exogenous effort model. For that purpose, we introduce some more notation. Consider a pair of coalitions $C_k, C_\ell \in \Pi$ and assume that some non-empty $T \subseteq C_\ell$ leaves C_ℓ and joins C_k, whereas all other coalitions remain the same. The resulting coalition structure is denoted by $\Pi^T_{C_\ell \to C_k}$.

Lemma 2. *In this exogenous effort model, assumptions 1-3 imply that for any* $C_k, C_\ell \in \Pi$ *and* $\Pi \in \mathbb{P}$ *there exist thresholds* $z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right)$ *and* $z_2\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right)$*, where* $\bar{\mathbf{Y}}_{-C_k} = \underbrace{\{1, 1..., 1\}}_{n-|C_k| \text{ times}}$*, such that*

(i) If $|C_k| \geq z_2\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right)$*, then the marginal increase in* C_k*'s winning probability by admitting players from* C_ℓ *is decreasing:*

$$p\left(C_k \cup \{j,k\}, \underset{C_\ell \to C_k}{\Pi}^{\{j,k\}}, \bar{\mathbf{Y}}\right) - p\left(C_k \cup \{j\}, \underset{C_\ell \to C_k}{\Pi}^{\{j\}}, \bar{\mathbf{Y}}\right) < p\left(C_k \cup \{j\}, \underset{C_\ell \to C_k}{\Pi}^{\{j\}}, \bar{\mathbf{Y}}\right) - p\left(C_k, \Pi, \bar{\mathbf{Y}}\right) \tag{3}$$

(ii) If $|C_k| \leq z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right)$*, then the marginal decrease in* C_k*'s winning probability by transferring players to* C_ℓ *is increasing:*

$$p\left(C_k \setminus \{i,j\}, \underset{C_k \to C_\ell}{\Pi}^{\{i,j\}}, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{i\}, \underset{C_k \to C_\ell}{\Pi}^{\{i\}}, \bar{\mathbf{Y}}\right) > p\left(C_k \setminus \{i\}, \underset{C_k \to C_\ell}{\Pi}^{\{i\}}, \bar{\mathbf{Y}}\right) - p(C_k, \Pi, \bar{\mathbf{Y}}) \tag{4}$$

Proof. See Appendix. □

Lemma 2 means that the marginal increase in the winning probability by adding members - through pair-wise transfers - is increasing as long as C_k's size is less than the threshold $z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right)$. Conversely, the marginal increase in the winning probability by adding members is decreasing if C_k's size is greater than the threshold $z_2\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right)$.[8] This property resonates with Assumption 2 where the winning probability displays increasing returns below a threshold, and then diminishing returns with effort. For this exogenous effort model, an increase in a coalition's effort (by adding members) is met with a simultaneous decrease in effort by another coalition (by transferring members). However, we show that the winning probability's convex-concave property extends to such discrete pair-wise transfers.

Based on the properties derived in Lemmas 1 and 2, we begin the equilibrium analysis for the coalition formation game described in section 2.2 for this exogenous effort model. Any proposer faces the following trade-off while maximising payoff (2): From part (B) in Lemma 1, larger coalitions have a greater worth. However, that worth is shared among more members. Hence, pair-wise transfers that increase the size of a coalition have an ambiguous effect on its member's payoff: the coalition worth increases, but individual share decreases with coalition size. Thus, the proposer chooses a coalition size to optimize the trade-off between coalition worth and size while accounting for the actions of rivals.

[8] Recall that by assumption we have $z_2\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right) > z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right)$. By continuity, there will exist a $\bar{z} \in \left[z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right), z_2\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right)\right]$ for which the marginal increase of the winning probability is increasing (resp. decreasing) for all sizes less (resp. greater) than $\bar{z}$.

Theorem 1. *In the exogenous effort model, if assumptions 1-3 are satisfied, then any stationary perfect equilibrium (SPE) is non-symmetric:* $\Pi^* = \{C_1^*, C_2^*, ...C_K^*\}$ *where* $\neq |C_j^*|$ *for at least one pair* $C_i^*, C_j^* \in \Pi^*$ *and* $K \in \{2, 3...n-1\}$.

Proof. Refer to the appendix. □

This result implies that if a standard CSF satisfies Assumption 1 - 3, then the equilibrium is an intermediate coalition structure. That is, neither the extreme of the grand coalition forms nor the other extreme of no coalitions occurs (as both are symmetric equilibria). In other words, the incentive to cooperate exists, but not full cooperation.

The intuition for this result is that Assumptions 2 and 3 lead to an increasing returns property: the marginal benefit from admitting members increases until a threshold. Assumption 2 introduces the incentive to cooperate, while assumption 3 relates this property to contest settings. At the same time, by forming the grand coalition there is no value addition from efficiency gains as all players divide the prize equally. Therefore, it is profitable to form a majority coalition to encash efficiency gains and engage in a contest.

Let us compare this result with those existing in literature. Using a certain specification of the Tullock contest success function, Skaperdas (1998) shows that in a three-player model and a generalized Tullock CSF, two players form a coalition if and only if the CSF displays a increasing returns property. Tan and Wang (2010) produce similar results where players form two competing coalitions at equilibrium with n players under the same increasing returns property in a stylised model. Theorem 1 generalizes this theory and identifies the economic principles under which the formation of multiple groups is possible. In the following result (Corollary 1.1), we show that the equilibrium is not limited to two coalitions, but three or more coalition may form depending on the nature of the winning probability function.

Corollary 1.1. *In the exogenous effort model, given assumptions 1-3, the formation of three or more coalitions is possible. The number of coalitions formed at equilibrium is likely to be high if*

(i) the threshold coalition size, $z_2\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}\right)$*, is low for any* $C_k \in \Pi$ *and* $\Pi \in \mathbb{P}$.

(ii) the marginal increase in winning probability diminishes rapidly beyond that threshold $z_2(.)$.

The sufficient conditions for the formation of exactly three coalitions at equilibrium are stated in the Appendix.

In the proof of this result, we first show that if the set of active players (those that have not yet agreed on forming coalitions) is non-empty, then splitting into two coalitions dominates forming a single coalition under certain conditions (Lemma 6). The discussion that follows this lemma in the Appendix relates these conditions with conditions (i) and (ii) in Corollary 1.1.

We state sufficient conditions for the formation of three coalitions in the Appendix. The intuition for this result is as follows. If $C_1 \subset N$ is the first coalition to form, then the remaining players, $N \backslash C_1$, may either split into further coalitions or form a single coalition. From Lemma 5, if the threshold $z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_1}\right)$ is high, then a single coalition $N \setminus C_1$ forms. However, if $z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_1}\right)$ is low and the increase in winning probability diminishes rapidly with size beyond $z_2\left(p(.), \Pi, \bar{\mathbf{Y}}_{-C_1}\right)$, then efficiency gains from forming, say $D \subset N \setminus C_1$, is higher than forming a single coalition $N \setminus C_1$.[9] Thus, three coalitions would tend to form. Note, that this reasoning is similar to that in Theorem 1 where we prove that atleast two coalitions form. Further, by induction, this logic extends to the formation of more than three coalitions. Lastly, we illustrate the formation of three coalitions in Example 1 with a 5 players.

Example 1. *Consider $n = 5$ player setting where the winning probability function is given by the generalised Tullock contest success function given by*

$$\bar{p}(C_k, \Pi, \bar{\mathbf{Y}}) = \frac{|C_k|^{R_{|C_k|}}}{|C_k|^{R_{|C_k|}} + \sum\limits_{j \neq k} |C_j|^{R_{|C_j|}}}. \tag{5}$$

If $R_2 > 1$ and $R_3 = R_4 = 2$, then the unique coalition structure formed at the stationary perfect equilibrium is

$$\Pi^* = \begin{cases} \{C_1^*, C_2^*, C_3^*\} \quad \textit{where} \quad |C_1^*| = 2, |C_2^*| = 2, |C_3^*| = 1 & \textit{if} \quad R_2 > 3 \\ \{D_1^*, D_2^*\} \quad \textit{where} \quad |D_1^*| = 4, |D_2^*| = 1 & \textit{if} \quad 1.9069 \leq R_2 \leq 3 \\ \{S_1^*, S_2^*\} \quad \textit{where} \quad |S_1^*| = 3, |S_2^*| = 2 & \textit{if} \quad 1 < R_2 < 1.9069 \end{cases}$$

In the exogenous effort model, the aggregate effort of the coalition is given by its size. The parameter $R_{|C_k|}$ is the impact factor for any coalition of size $|C_k|$. In proposition 1 (section 4) we show that $\bar{p}(.)$ satisfies assumptions 1-3 as long as $R_{|C_k|} > 1$ for any $C_k \in \Pi$ and $\Pi \in \mathbb{P}$. Therefore, we have assumed $R_i > 1$ for all $i \in \{2, 3, 4\}$. Thus, the formation of three coalitions at equilibrium is demonstrated.

Summarizing this section, we introduced the exogenous effort assumption and studied the coalition formation game discussed in Section 2.2. We find that if a contest success function that satisfies standard axioms (A1) also displays convex/concavity (A2) and sub/supermodularity (A3), then players form coalitions, but not the grand coalition. Further, the possibility of forming three or more coalitions exists.

[9] As $z_2(.) \geq z_1(.)$, low $z_2(.)$ implies low $z_1(.)$.

4 Endogenous effort model

In the previous section 3, we analyzed the model assuming efforts are constant. In this section, we relax that assumption and revisit the model with endogenous efforts discussed in section 2. The endogenous effort model involves a two-stage game: at stage one, coalitions form; at stage two, players choose efforts. The results from section 3 apply to endogenous efforts provided the cost function is sufficiently convex. We observe this continuity in results because the exogenous effort is a sub-case of a convex cost function. After all, the cost of increased effort from zero to one is negligible, and then increasing effort beyond one has an infinite cost. Hence, by continuity, our results extend to sufficiently convex cost functions.

To demonstrate this argument, we first discuss the equilibrium of effort at stage 2. For simplicity, we first assume an effort-independent equal division rule. Thus, we include the free-rider effect, but at the same time, exclude the complications that arise from the effort-dependent rules. [10] Second, we conduct this analysis with the generalized Tullock CSF and exponential cost to keep the analysis tractable. Thus, the individual payoff for the effort-independent equal division rule is given by

$$u(C_k, \Pi, \mathbf{Y}) = \frac{\bar{p}(C_k, \Pi, \mathbf{Y})}{|C_k|} - y_i^{\alpha} \quad \text{where} \quad \alpha > 1.$$

The winning probability in the payoff above is given by

$$\bar{p}(C_k, \Pi, \mathbf{Y}) = \frac{Y_{C_k}^{R_{C_k}}}{Y_{C_k}^{R_{C_k}} + \sum_{C_j \neq C_k} Y_{C_j}^{R_{C_j}}}. \tag{6}$$

The parameter R_{C_k} quantifies the impact of coalition C_k's aggregate effort on its winning probability. We first identify the class of generalised Tullock CSFs characterised by the assumptions stated in Section 2.

Proposition 1. *The generalized Tullock contest success function (6) satisfies assumptions 1-3 if and only if $R_{C_k} > 1$ for all $C_k \in \Pi$.*

The impact factor $R_{C_k} > 1$ implies that coalitions generate efficiency gains from cooperation. This specification accommodates settings where the efficiency gains generated decreases (or increases) with coalition size. Thus, our assumptions identify a wide class of contest success functions, specifically all those settings where cooperation generates efficiency gains.

[10] As players are symmetric, members of any coalition will exert equal efforts at equilibrium. We prove this for the effort independent sharing rule in the proof of proposition 2. The same argument extends to effort-dependent sharing rules. Hence, this assumption does not change the economic insights developed in this section.

Next, we analyse the effort equilibrium. The cost function is strictly convex for $\alpha > 1$. Hence, from assumption 2 we conclude that for any given coalition structure and aggregate rivals' efforts, the payoff function is convex-concave in y_i. Therefore, the existence of a unique equilibrium of efforts is guaranteed. However, players may exert zero effort at equilibrium (a corner solution). If all the members of a coalition exert zero effort at equilibrium, then that coalition fails to participate in the contest. In other words, for any coalition structure, the contest is played among those coalitions where at least one member chooses to exert positive effort. To ensure participation of all players, we provide a sufficient condition for exerting positive effort.

Proposition 2. *If $\alpha > R_{|C_k|} + 1$ for all $C_k \in \Pi$, then there always exists an interior equilibrium of efforts: $Y^*_{C_k} > 0$ for any $C_k \in \Pi$ and $\Pi \in \mathbb{P}$.*

This means that irrespective of the coalition structure formed, every coalition participates in the contest if the cost function is sufficiently convex. The reason we focus on this situation is because - as explained in the first paragraph of this section - the main results from Section 3 are applicable to sufficiently convex functions. Therefore, we restrict our attention to this situation for the remaining section to avoid the complications that may arise from potential corner solutions.

Our next result states the condition for which our main result (theorem 1) extends to the case of endogenous effort.

Proposition 3. *If the payoff function is given by (6) such that $R_{C_k} = R$ for all $C_k \in \Pi$ and $\Pi \in \mathbb{P}$, then the equilibrium partition is non-symmetric if $\alpha > \max\{\bar{\alpha}, R+1\}$ where $\bar{\alpha} \in (2, \infty)$ satisfies*

$$\frac{\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\bar{\alpha}}\right)}}{|C|\left(1+\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\bar{\alpha}}\right)}\right)}\left[1-\frac{R}{\bar{\alpha}|C|\left(1+\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\bar{\alpha}}\right)}\right)}\right]=\frac{1}{n}.$$

This proposition states that the equilibrium coalition structure will always be non-symmetric as long as α is sufficiently large. Note that the larger α is, the closer is the endogenous effort model to the exogenous effort model discussed in section 3. Hence, we prove our initial conjecture that the results from section 3 extend to endogenous budgets when the cost function is sufficiently convex.

The proof of this proposition follows two steps. First, we prove that the grand coalition delivers a higher individual payoff compared to any equal-sized coalition structure. This is because any individual player's expected share of the prize in an equal-sized coalition structure is equal to that obtained by forming the grand coalition. However, in the presence of competition, members need to bear the cost of exerting effort that lowers their payoff. Second, we show that for any non-symmetric bipartite coalition structure, the members of the larger coalition earn a payoff that exceeds the grand coalition if α is sufficiently high. Lastly, we show that if the size of the larger coalition is sufficiently high, then the optimal response of the remaining players is to band together and form a coalition. Hence, for a sufficiently large α, there exists a bipartite structure that strictly dominates forming the grand coalition. Note that we have assumed additional symmetry of the impact factors, $R_{C_k} = R$, to prove this result. However, by continuity, this would apply even to asymmetric impact factors.

5 Conclusion

In this paper, we analyze a group contest where identical players endogenously form coalitions. The coalition formation mechanism operates under the condition that the agreements are binding; i.e., players cannot deviate once a coalition is formed. This mechanism, based on Bloch(1996) and Ray and Vohra (1997), is essentially a game among the players whose outcome is a coalition structure, and the solution concept is defined for a coalition structure, based on the players' strategies in that mechanism.

The main novelty of our model is that we formulate the partition function as a contest success function through certain assumptions. These assumptions fit group contest settings where cooperation leads to efficiency gains. We show that if the contest success function is specified through those assumptions, then any equilibrium coalition structure consists of two or more coalitions where no two coalitions are of equal size. This indicates that there is an imbalance of power across coalitions at equilibrium. Further, we show that the formation of three or more coalitions is possible at equilibrium. As literature only provides the conditions for the formation of two competing coalitions, this result provides new insight into the literature.

6 Appendix

Proof of Lemma 1

Proof. The unit effort vector is $\bar{\mathbf{Y}}$ where the effort exerted by any player $y_i = 1$ for any $i \in N$. Now, (P1) and (P3) are a direct implication of assumption (1A) and (1C). To show (P2), consider two players $i \in C_k$ and $j \in C_\ell$ where C_ℓ is some rival coalition in Π. Let $\mathbf{Y}_{j=b}^{i=a}$ denote the effort vector where $y_i = a$, $y_j = b$ and $y_k = 1$ for all $k \neq i, j$.

By assumption (1B) we get

$$p\left(C_k,\Pi,\mathbf{Y}_{j=0}^{i=2}\right) \;>\; p\left(C_k,\Pi,\mathbf{Y}_{j=1}^{i=2}\right) > p\left(C_k,\Pi,\mathbf{Y}_{j=1}^{i=1}\right) \tag{7}$$

$$p\left(C_\ell,\Pi,\mathbf{Y}_{j=1}^{i=1}\right) \;>\; p\left(C_\ell,\Pi,\mathbf{Y}_{j=0}^{i=1}\right) > p\left(C_\ell,\Pi,\mathbf{Y}_{j=0}^{i=2}\right). \tag{8}$$

Observe that $\mathbf{Y}_{j=1}^{i=1} = \bar{\mathbf{Y}}$, therefore

$$p\left(C_k,\Pi,\mathbf{Y}_{j=0}^{i=2}\right) = p\left(C_k \cup \{j\}, \underset{C_\ell \to C_k}{\Pi^{\{j\}}}, \bar{\mathbf{Y}}\right) \quad \text{and} \quad p\left(C_\ell,\Pi,\mathbf{Y}_{j=0}^{i=2}\right) = p\left(C_\ell \setminus \{j\}, \underset{C_\ell \to C_k}{\Pi^{\{j\}}}, \bar{\mathbf{Y}}\right)$$

In words, an increase in unit effort by player $i \in C_k$, and a simultaneous decrease in effort by player $j \in C_\ell$ is equivalent to transferring player j from coalition C_ℓ to C_k. From inequalities (7) and (8) we obtain

$$p\left(C_k \cup \{j\}, \underset{C_\ell \to C_k}{\Pi}^{\{j\}}, \bar{\mathbf{Y}}\right) > p\left(C_k,\Pi,\bar{\mathbf{Y}}\right) \quad \text{and} \quad p\left(C_\ell \setminus \{j\}, \underset{C_\ell \to C_k}{\Pi}^{\{j\}}, \bar{\mathbf{Y}}\right) < p\left(C_\ell,\Pi,\bar{\mathbf{Y}}\right). \tag{9}$$

Next, consider a transfer of some $T \subseteq C_k$ to C_ℓ where $|T| = \frac{|C_k|-|C_\ell|}{2}$ and $|C_k| > |C_\ell|$. As $|C_k|-|T| = |C_\ell| + |T|$, symmetry entails $p\left(C_k \setminus T, \underset{C_k \to C_\ell}{\Pi^T}, \bar{\mathbf{Y}}\right) = p\left(C_\ell \cup T, \underset{C_k \to C_\ell}{\Pi^T}, \bar{\mathbf{Y}}\right)$. Thus, by inequality (9) we get

$$p(C_k,\Pi,\bar{\mathbf{Y}}) \geq p\left(C_k \setminus T, \underset{C_k \to C_\ell}{\Pi^T}, \bar{\mathbf{Y}}\right) = p\left(C_\ell \cup T, \underset{C_k \to C_\ell}{\Pi^T}, \bar{\mathbf{Y}}\right) \geq p(C_\ell,\Pi,\bar{\mathbf{Y}}).$$

Hence, (P2) is proved. □

To prove the proposition 2, we introduce the following lemma. Note that this lemma is proved for the case where effort is variable.

Lemma 3.

$$\frac{\partial^2 p(C_k,\Pi,\mathbf{Y})}{\partial y_i^2} \lesseqgtr 0 \quad \textit{for all} \quad Y_{C_k} \lesseqgtr z_1(p(.),\Pi,Y_{-C_k})$$

where $i \in C_j$, $C_j \neq C_k$ *and* $z_1(p(.),\Pi,Y_{-C_k})$ *is the threshold from Assumption 2.*

Proof. For any $\Pi = \{C_1, C_2 ..., C_K\}$, the winning probability of all coalitions sums to one (Assumption 1A).

$$\sum_{i=1}^{K} p(C_i,\Pi,\mathbf{Y}) = 1. \tag{10}$$

Note that the effort vector is variable: $\mathbf{Y} = \{y_1, y_2 ..., y_n\}$. As the partition function is continuous and differentiable with respect to the effort of any player, we take the double derivative of equality (10)

$$\frac{d^2 p(C_k,\Pi,\mathbf{Y})}{dy_i^2} + \sum_{j\neq k} \frac{d^2 p(C_j,\Pi,\mathbf{Y})}{dy_i^2} = 0 \quad \text{where} \quad i \in C_k. \tag{11}$$

By symmetry, the sign, either positive or negative, of the partition function's second order derivative for any two coalitions apart from C_k must be identical.

$$sign\left|\frac{d^2p(C_j,\Pi,\mathbf{Y})}{dy_i^2}\right| = sign\left|\frac{d^2p(C_\ell,\Pi,\mathbf{Y})}{dy_i^2}\right| \quad \text{for any} \quad C_j, C_\ell \in \Pi \quad \text{and} \quad j \neq \ell \neq k.$$

Now, in order for the equality (11) to hold it must be the case that

$$sign\left|\frac{d^2p(C_k,\Pi,\mathbf{Y})}{dy_i^2}\right| = -sign\left|\frac{d^2p(C_j,\Pi,\mathbf{Y})}{dy_i^2}\right| \quad \text{for any} \quad C_j \in \Pi \quad \text{and} \quad j \neq k.$$

Therefore if $\frac{d^2p(C_k,\Pi,\mathbf{Y})}{dy_i^2} \gtreqless 0$, then $\frac{d^2p(C_j,\Pi,\mathbf{Y})}{dy_i^2} \lesseqgtr 0$ for any $j \neq k$. By assumption 2 we know that the partition function is convex (resp. concave) or $\frac{d^2p(C_k,\Pi,\mathbf{Y})}{dy_i^2} > \; (resp. <)\; 0$ whenever $Y_{C_k} < \; (resp. >)\; z_1(p(.),\Pi,Y_{-C_k})$. Hence, proved. □

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Proof of Lemma 2

Proof. Consider the exogenous effort model where each player exerts constant effort $y_i = 1$ for any $i \in N$ and the unit effort vector is $\bar{\mathbf{Y}}$. Therefore, $Y_{C_k} = |C_k|$ for any $C_k \in \Pi$. Consider a different coalition $C_\ell \in \Pi$ such that $|C_\ell| \geq 2$.

Part I: *To show that admitting members to coalition C_k leads to a diminishing increase in its winning probability stated in inequality* (3).

Let $|C_k| \geq z_2(p(.),\Pi,\bar{\mathbf{Y}}_{-C_k}) \geq z_1(p(.),\Pi,\bar{\mathbf{Y}}_{-C_k})$ as necessitated by Assumption 2. Also, consider players $i \in C_k$ and $j,k \in C_\ell$ such that $\mathbf{Y}^{i=a}_{j=b,k=c}$ denotes the effort vector where $y_i = a$, $y_j = b$, $y_k = c$ and $y_\ell = 1$ for all $\ell \neq i,j,k$. Note that $\bar{\mathbf{Y}} = \mathbf{Y}^{i=1}_{j=1,k=1}$.

Step 1: To show that

$$p\left(C_k,\Pi,\mathbf{Y}^{i=2}_{j=1,k=1}\right) - p\left(C_k,\Pi,\mathbf{Y}^{i=1}_{j=1,k=1}\right) > p\left(C_k,\Pi,\mathbf{Y}^{i=3}_{j=0,k=1}\right) - p\left(C_k,\Pi,\mathbf{Y}^{i=2}_{j=0,k=1}\right). \tag{12}$$

From assumption 2, we know that the winning probability of coalition C_k displays marginal decreasing returns with its effort. That is, if the effort of player $i \in C_k$ increases, while that of players $j,k \in C_\ell$ remains unchanged, then we get

$$p\left(C_k,\Pi,\mathbf{Y}^{i=2}_{j=1,k=1}\right) - p\left(C_k,\Pi,\mathbf{Y}^{i=1}_{j=1,k=1}\right) > p\left(C_k,\Pi,\mathbf{Y}^{i=3}_{j=1,k=1}\right) - p\left(C_k,\Pi,\mathbf{Y}^{i=2}_{j=1,k=1}\right) \tag{13}$$

By assumption 3, the winning probability of coalition C_k displays supermodularity with respect to the effort of a rival coalition C_j. That is, if the effort of player $j \in C_\ell$ increases, then the marginal benefit accrued through player i's increase in effort increases.

$$p\left(C_k,\Pi,\mathbf{Y}^{i=3}_{j=1,k=1}\right) - p\left(C_k,\Pi,\mathbf{Y}^{i=2}_{j=1,k=1}\right) > p\left(C_k,\Pi,\mathbf{Y}^{i=3}_{j=0,k=1}\right) - p\left(C_k,\Pi,\mathbf{Y}^{i=2}_{j=0,k=1}\right) \tag{14}$$

From equations (13) and (14) we obtain inequality (12)

Step 2: To show that

$$p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=1}\right) - p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=1,k=1}\right) > p\left(C_k, \Pi, \mathbf{Y}^{i=3}_{j=0,k=0}\right) - p\left(C_k, \Pi, \mathbf{Y}^{i=3}_{j=0,k=1}\right). \tag{15}$$

As $|C_k| \geq z_1(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k})$, from lemma 3, coalition C_k's winning probability displays increasing marginal returns with any rival C_j's effort. Therefore, we have

$$p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=1}\right) - p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=1,k=1}\right) > p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=0}\right) - p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=1}\right). \tag{16}$$

Again, by assumption 3, if the effort of player $k \in C_\ell$ increases, then the marginal benefit accrued through player i's increase in effort increases.

$$p\left(C_k, \Pi, \mathbf{Y}^{i=3}_{j=0,k=1}\right) - p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=1}\right) > p\left(C_k, \Pi, \mathbf{Y}^{i=3}_{j=0,k=0}\right) - p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=0}\right).$$

Rewriting the above inequality gives

$$p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=0}\right) - p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=1}\right) > p\left(C_k, \Pi, \mathbf{Y}^{i=3}_{j=0,k=0}\right) - p\left(C_k, \Pi, \mathbf{Y}^{i=3}_{j=0,k=1}\right). \tag{17}$$

From equations (16) and (17) we obtain inequality (15).

Step 3: Add inequalities (12) and (15) mentioned in steps 1 and 3 respectively to obtain

$$p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=1}\right) - p\left(C_k, \Pi, \mathbf{Y}^{i=1}_{j=1,k=1}\right) > p\left(C_k, \Pi, \mathbf{Y}^{i=3}_{j=0,k=0}\right) - p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=1}\right). \tag{18}$$

Observe $p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=1}\right)$ is equivalent to the winning probability obtained by transferring player $j \in C_\ell$ to C_k in the exogenous effort model.

$$p\left(C_k, \Pi, \mathbf{Y}^{i=2}_{j=0,k=1}\right) \equiv p\left(C_k \cup \{j\}, \underset{C_\ell \to C_k}{\Pi}^{\{j\}}, \bar{\mathbf{Y}}\right) \tag{19}$$

Similarly, $p\left(C_k, \Pi, \mathbf{Y}^{i=3}_{j=0,k=0}\right)$ is equivalent to the winning probability obtained by transferring players $j, k \in C_\ell$ to C_k.

$$p\left(C_k, \Pi, \mathbf{Y}^{i=3}_{j=0,k=0}\right) \equiv p\left(C_k \cup \{j,k\}, \underset{C_\ell \to C_k}{\Pi}^{\{j,k\}}, \bar{\mathbf{Y}}\right) \tag{20}$$

Hence, inequality (18) is equivalent to (3).

Part II: *To show that admitting members to coalition C_k displays an increase returns property as stated in inequality* (4).

Let $|C_k| \leq z_1(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}) \leq z_2(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k})$. Also, consider players $i, j \in C_k$ and $k \in C_\ell$ such that $\mathbf{Y}^{i=a,j=b}_{k=c}$ denotes the effort vector where $y_i = a$, $y_j = b$, $y_k = c$ and $y_\ell = 1$ for all $\ell \neq i, j, k$.

Note that $\bar{\mathbf{Y}} = \mathbf{Y}_{k=1}^{i=1,j=1}$.

Step 1: To show

$$p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=1,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=0,j=1}\right) > p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=0}\right). \tag{21}$$

From assumption 2 , the winning probability of coalition C_k displays increasing marginal returns with its own effort. That is, if the effort of players $i, j \in C_k$ increases, while that of player $k \in C_\ell$ remains unchanged, then we get

$$p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=1,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=0,j=1}\right) > p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=0,j=0}\right). \tag{22}$$

Note that the RHS of the above inequality is the increase in winning probability obtained by a unit increase in effort by player $j \in C_k$. Consequently, the aggregate effort of coalition C_k increases from $|C_k| - 2$ to $|C_k| - 1$. The LHS is the increase in winning probability obtained by a unit increase in effort by player $i \in C_k$ given player j's increase. Hence, the aggregate effort of coalition C_k increases from $|C_k| - 1$ to $|C_k|$.

By assumption 3, the winning probability of coalition C_k displays submodularity concerning the effort of a rival coalition C_j. That is, if the effort of player $k \in C_\ell$ increases, then the marginal benefit accrued through player j's increase in effort decreases.

$$p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=0,j=0}\right) > p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=0}\right). \tag{23}$$

From equations (22) and (23) we obtain inequality (21).

Step 2: To show

$$p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right) > p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=0}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=3}^{i=0,j=0}\right). \tag{24}$$

As $|C_k| \leq z_1(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k})$, from lemma 3, the winning probability of coalition C_k displays diminishing marginal returns with player k's effort who belongs to the rival coalition C_ℓ.

$$p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=0,j=1}\right) > p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=3}^{i=0,j=1}\right). \tag{25}$$

Again from assumption 3 we can write

$$p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=0}\right) > p\left(C_k, \Pi, \mathbf{Y}_{k=3}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=3}^{i=0,j=0}\right).$$

Rewriting the above inequality gives

$$p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=3}^{i=0,j=1}\right) > p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=0}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=3}^{i=0,j=0}\right). \tag{26}$$

From equations (25) and (26) we obtain (24).

Step 3: Add inequalities (21) and (24) from steps 1 and 2 respectively to obtain (4)

$$p\left(C_k, \Pi, \mathbf{Y}_{k=1}^{i=1,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right) > p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right) - p\left(C_k, \Pi, \mathbf{Y}_{k=3}^{i=0,j=0}\right). \tag{27}$$

Observe that $p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right)$ is equivalent to the winning probability obtained by transferring player i from C_k to C_ℓ.

$$p\left(C_k, \Pi, \mathbf{Y}_{k=2}^{i=0,j=1}\right) \equiv p\left(C_k \setminus \{i\}, \underset{C_k \to C_\ell}{\Pi}^{\{i\}}, \bar{\mathbf{Y}}\right) \tag{28}$$

Similarly, $p\left(C_k, \Pi, \mathbf{Y}_{k=3}^{i=0,j=0}\right)$ is equivalent to the winning probability obtained by transferring players i, j from C_k to C_ℓ.

$$p\left(C_k, \Pi, \mathbf{Y}_{k=3}^{i=0,j=0}\right) \equiv p\left(C_k \setminus \{i,j\}, \underset{C_k \to C_\ell}{\Pi}^{\{i,j\}}, \bar{\mathbf{Y}}\right) \tag{29}$$

Hence, inequality (18) is equivalent to (4).

□

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We prove the main result in theorem 1, through a sequence of smaller results stated in lemmas 4 and 5. This structure should facilitate the reader to connect the statements of these lemmas to understand the main intuition for the result.

Lemma 4. *If* $|C_k| \leq z_1(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k})$*, then admitting members increases its members' payoff. Formally,*

$$\frac{p\left(C_k, \Pi, \bar{\mathbf{Y}}\right)}{|C_k|} > \frac{p\left(C_k \setminus \{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}}, \bar{\mathbf{Y}}\right)}{|C_k| - 1}$$

Proof. The individual payoff for members in C_k can be algebrically manipulated and written as follows.

$$\frac{p\left(C_k, \Pi, \bar{\mathbf{Y}}\right)}{|C_k|} = \frac{\left[p\left(C_k, \Pi, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}}, \bar{\mathbf{Y}}\right)\right] + p\left(C_k \setminus \{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}}, \bar{\mathbf{Y}}\right)}{|C_k|} \tag{30}$$

Next, suppose $i \in C_k$ is transferred to C_j. Subtract the payoff of members in the coalition $C_k \setminus \{i\}$, that is generated as a consequence of the transfer, from (30).

$$\frac{p\left(C_k,\Pi,\bar{\mathbf{Y}}\right)}{|C_k|} - \frac{p\left(C_k\setminus\{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}},\bar{\mathbf{Y}}\right)}{|C_k|-1} = \frac{(|C_k|-1)\left[p\left(C_k,\Pi,\bar{\mathbf{Y}}\right)-p\left(C_k\setminus\{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}},\bar{\mathbf{Y}}\right)\right]-p\left(C_k\setminus\{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}},\bar{\mathbf{Y}}\right)}{|C_k|(|C_k|-1)} \tag{31}$$

In the remaining proof, we show that the numerator of the expression on the RHS of equality (31) is positive due to the convexity assumption 2 that results in inequality (4) stated in Lemma 2 for the exogenous effort model.

Step 1: To show that inequality (4) holds for coalitions smaller than C_k.

As we have assumed $z_1(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k})$ to decrease with effort in our model, for any possibly empty set $T \subseteq C_k$, we have

$$z_1(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k}) \leq z_1\left(p(.), \underset{C_k \to C_j}{\Pi^T}, \bar{\mathbf{Y}}_{-\{C_k \setminus T\}}\right) \quad \text{as} \quad \bar{\mathbf{Y}}_{-\{C_k \setminus T\}} \leq \bar{\mathbf{Y}}_{-C_k}.$$

$$\text{Therefore} \quad |C_k \setminus T| \leq z_1\left(p(.), \underset{C_k \to C_j}{\Pi^T}, \bar{\mathbf{Y}}_{-\{C_k \setminus T\}}\right) \quad \text{for all} \quad T \subseteq C_k$$

Given $|C_k| \leq z_1(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k})$, from inequality (4) in lemma 2 we have

$$p\left(C_k \setminus T, \underset{C_k \to C_j}{\Pi^T}, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{T \cup \{i\}\}, \underset{C_k \to C_j}{\Pi^{T \cup \{i\}}}, \bar{\mathbf{Y}}\right) > p\left(C_k \setminus \{T \cup \{i\}\}, \underset{C_k \to C_j}{\Pi^{T \cup \{i\}}}, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{T \cup \{i,j\}\}, \underset{C_k \to C_j}{\Pi^{T \cup \{i,j\}}}, \bar{\mathbf{Y}}\right) \tag{32}$$

As the above inequality holds for any $T \subseteq C_k$, we conclude that if $|C_k| \leq z_1(p(.), \Pi, \bar{\mathbf{Y}}_{-C_k})$, then inequality (4) also holds for any coalition size smaller than C_k.

Step 2: Suppose we begin with a singleton coalition, and continue to admit members in unit increments till the coalition C_k forms. We show that the terminal increase in winning probability given by the LHS of (33) is greater than the average of all increases in transfers to coalition C_k.

As inequality (32) holds for any $T \subseteq C_k$, we get

$$p\left(C_k, \Pi, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}}, \bar{\mathbf{Y}}\right) > p\left(C_k \setminus \{T \cup \{i\}\}, \underset{C_k \to C_j}{\Pi^{T \cup \{i\}}}, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{T \cup \{i,j\}\}, \underset{C_k \to C_j}{\Pi^{T \cup \{i,j\}}}, \bar{\mathbf{Y}}\right) \quad \forall T \subset C_k. \tag{33}$$

Next, we generate a system of inequalities by increasing the set of players being transferred T as follows

$$\begin{aligned}
p\left(C_k, \Pi, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}}, \bar{\mathbf{Y}}\right) &> p\left(C_k \setminus \{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}}, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{i,j\}, \underset{C_k \to C_j}{\Pi^{\{i,j\}}}, \bar{\mathbf{Y}}\right) \quad \text{where} \quad T = \{\emptyset\} \\
p\left(C_k, \Pi, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}}, \bar{\mathbf{Y}}\right) &> p\left(C_k \setminus \{i,k\}, \underset{C_k \to C_j}{\Pi^{\{i,k\}}}, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{i,j,k\}, \underset{C_k \to C_j}{\Pi^{\{i,j,k\}}}, \bar{\mathbf{Y}}\right) \quad \text{where} \quad T = \{k\} \\
\vdots &> \vdots \\
p\left(C_k, \Pi, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}}, \bar{\mathbf{Y}}\right) &> p\left(\{\ell\}, \underset{C_k \to C_j}{\Pi^{C_k \setminus \{\ell\}}}, \bar{\mathbf{Y}}\right) - p\left(\{\emptyset\}, \underset{C_k \to C_j}{\Pi^{\{C_k\}}}, \bar{\mathbf{Y}}\right) \quad \text{where} \quad T = C_k \setminus \{\ell\}
\end{aligned}$$

Adding the above inequalities, we get

$$p\left(C_k, \Pi, \bar{\mathbf{Y}}\right) - p\left(C_k \setminus \{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}}, \bar{\mathbf{Y}}\right) > \frac{p\left(C_k \setminus \{i\}, \underset{C_k \to C_j}{\Pi^{\{i\}}}, \bar{\mathbf{Y}}\right) - p\left(\{\emptyset\}, \underset{C_k \to C_j}{\Pi^{\{C_k\}}}, \bar{\mathbf{Y}}\right)}{|C_k| - 1}. \tag{34}$$

As $p\left(\{\emptyset\},\Pi^{\{C_k\}}_{C_k\to C_j},\bar{\mathbf{Y}}\right)=0$, from equality (31) and inequality (34), we conclude that

$$\frac{p\left(C_k,\Pi,\bar{\mathbf{Y}}\right)}{|C_k|}-\frac{p\left(C_k\setminus\{i\},\Pi^{\{i\}}_{C_k\to C_j},\bar{\mathbf{Y}}\right)}{|C_k|-1}>0$$

□

The next lemma shows that if the number of active players is less than its respective threshold, then all the remaining players form a coalition. For this purpose, we restate the notation defined in section 2.2. Consider a sub-coalition structure at round $t\geq 0$ denoted by $\pi_t=\{C_1,C_2,...C_t\}$ such that $K(\pi_t)=N\setminus A_t$ where $A_t\neq\emptyset$ is the set of active players.[11] The final coalition structure will be $\Pi=\pi_t\cup\{C_{t+1},C_{t+2}...C_K\}$ where $A_t=\bigcup\limits_{i=1}^{K-t}C_{t+i}$.

Lemma 5. *If $z_1\left(p(.),\Pi,\bar{\mathbf{Y}}_{-A_t}\right)\geq|A_t|$, then the optimal strategy of the proposer is $\sigma_i^P(\pi_t,N)=A_t$ and the responder is $\sigma_i^R(\pi_t,N,f(\pi_t))=Y$.*

Proof. The central theme of this proof is that members of coalitions C_i and C_j where $i,j\geq t+1$ always benefit from merging. Therefore, given the sub-coalition structure at round $t\geq 0$ is $\pi_t=\{C_1,C_2,...C_t\}$, the optimal proposal at round $t+1$ is $\sigma_i^P(\pi_t,N)=A_t$.

To show this, we divide our proof in two steps. Suppose the coalition structure $\Pi=\pi_t\cup\{C_{t+1},C_{t+2}...C_K\}$ forms.

Step 1: To show that $u_i(C_i,\Pi,\bar{\mathbf{Y}})>u_i(C_j,\Pi,\bar{\mathbf{Y}})$ if $|C_i|>|C_j|$ for any $i,j\geq t+1$.

Without loss of generality, let $|C_i|>|C_j|$ where $i,j\geq t+1$. Transfer a subset of players $S\subset C_i$ to C_j. As $|A_t|>|C_i|$, from Lemma 4 and symmetry we can conclude that for $|S|=\frac{|C_i|-|C_j|}{2}$,

$$\frac{p\left(C_i,\Pi,\bar{\mathbf{Y}}\right)}{|C_i|}>\frac{p\left(C_i\setminus S,\Pi^{S}_{C_i\to C_j},\bar{\mathbf{Y}}\right)}{|C_i|-|S|}=\frac{p\left(C_j\cup S,\Pi^{S}_{C_i\to C_j},\bar{\mathbf{Y}}\right)}{|C_j|+|S|}>\frac{p\left(C_j,\Pi,\bar{\mathbf{Y}}\right)}{|C_j|}\tag{35}$$

Step 2: To show that $u_i(C_i\cup C_j,\Pi^{C_j}_{C_j\to C_i},\bar{\mathbf{Y}})>\max\{u_i(C_i,\Pi,\bar{\mathbf{Y}}),u_i(C_j,\Pi,\bar{\mathbf{Y}})\}$ for any $i,j\geq t+1$.

Consider a sequential transfer of players form C_j to C_i. As $|A_t|\geq|C_i|+|C_j|$, from Lemma 4 and inequality (35), we can write

$$\frac{p\left(C_i\cup C_j,\Pi^{C_j}_{C_j\to C_i},\bar{\mathbf{Y}}\right)}{|C_i|+|C_j|}>...>\frac{p\left(C_i\cup\{i,j\},\Pi^{\{i,j\}}_{C_j\to C_i},\bar{\mathbf{Y}}\right)}{|C_i|+2}>\frac{p\left(C_i\cup\{i\},\Pi^{\{i\}}_{C_j\to C_i},\bar{\mathbf{Y}}\right)}{|C_i|+1}>\frac{p\left(C_i,\Pi,\bar{\mathbf{Y}}\right)}{|C_i|}>\frac{p\left(C_j,\Pi,\bar{\mathbf{Y}}\right)}{|C_j|}$$

[11] Note that $\pi_0=\emptyset$.

Hence, by merging coalitions C_i and C_j yields the maximum individual payoff to the members.

$$\frac{p\left(C_i \cup C_j, \underset{C_j \to C_i}{\Pi^{C_j}}, \bar{\mathbf{Y}}\right)}{|C_i| + |C_j|} > \max\left\{\frac{p\left(C_i, \Pi, \bar{\mathbf{Y}}\right)}{|C_i|}, \frac{p\left(C_j, \Pi, \bar{\mathbf{Y}}\right)}{|C_j|}\right\} \quad \text{for any} \quad i, j \geq t+1.$$

Therefore, the proposer at $t+1$ will never propose $\sigma_i^P(\pi_t, N) = C_i$ where $C_i \subset A_t$ because merging with any other coalition $C_j \subset A_t$ always yields a higher individual payoff. Also, the responders will never accept C_i because they can reject it and do better by proposing A_t. Hence, proved. □

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Proof of Theorem 1 :

Proof. This proof is divided into two parts.

Part I: Existence of Equilibrium

The symmetric game of sequential coalition formation described in section 2.2 is a finite game of perfect information with perfect recall. Hence, it admits a subgame perfect equilibrium in pure strategies, and therefore, the existence of equilibrium is guaranteed.

Part II: The Equilibrium is Non-Symmetric

Let $\bar{\Pi} = \{S_1, S_2\}$ denote a bipartite coalition structure where $|S_2| = n - |S_1|$. The central idea behind this proof is to show that there always exists a bipartite coalition structure that dominates the formation of any symmetric equilibrium.

Step 1: We show that the upper bound for the threshold is $z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_i}) < n$ in the exogenous effort model where $i \in \{1, 2\}$.

To show this, consider the contrapositive. That is, suppose $z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_i}) \geq n$. Without loss of generality, let $|S_1| \geq |S_2|$. Next, transfer any subset of players $T \subseteq S_1$ to S_2 such that $|T| = |S_2|$.

As $|S_2| + |T| \leq n$ and $z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2 \cup T}) \geq n$, we get $|S_2| + |T| \leq z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2 \cup T})$. Hence, from lemma 2, we obtain the following inequality.

$$p(S_2 \cup T, \underset{S_1 \to S_2}{\bar{\Pi}^T}, \bar{\mathbf{Y}}) - p(S_2, \bar{\Pi}, \mathbf{Y}) > p(S_2, \bar{\Pi}, \bar{\mathbf{Y}}) - p(\emptyset, \underset{S_2 \to S_1}{\bar{\Pi}^{S_2}}, \bar{\mathbf{Y}}) \quad \text{where} \quad |T| = |S_2| \tag{36}$$

If $|S_2| = \frac{n}{2}$, then $|S_1| = n - |S_2| = \frac{n}{2}$. Therefore, symmetry yields $p(S_2, \bar{\Pi}, \bar{\mathbf{Y}}) = \frac{1}{2}$. Substituting $p(S_2, \bar{\Pi}, \bar{\mathbf{Y}}) = \frac{1}{2}$ and $p(\emptyset, \underset{S_2 \to S_1}{\bar{\Pi}^{S_2}}, \bar{\mathbf{Y}}) = 0$ in (36), we get

$$p(N, \underset{S_1 \to S_2}{\bar{\Pi}^T}, \bar{\mathbf{Y}}) > 1 \quad \text{where} \quad N = S_2 \cup T \quad \text{is the grand coalition.}$$

However, this violates assumption 1A. Therefore, $z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-N}) < n$. As the threshold decreases with rival's effort, we have $z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2}) \leq z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-N}) < n$ for any $|S_2| \in \{1, 2..., n\}$.

Step 2: To show that there exists a fixed point such that $|S_2^*| = z_1(p(.), \Pi, \bar{\mathbf{Y}}_{-S_2^*})$ where $|S_2^*| \in \{1, 2.., n-1\}$.

As the coalition structure $\bar{\Pi}$ is bipartite, we know that $|S_1| = n - |S_2|$. Therefore, as the size of S_1 increases, the size of S_2 decreases.

From the primary assumptions, we know that $z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2})$ decreases with an increases in rival coalitions efforts $\bar{\mathbf{Y}}_{-S_2}$. In this exogenous effort model, as the coalition structure is bipartite, $\bar{\mathbf{Y}}_{-S_2}$ is an $|S_1|$-dimensional unit vector. Therefore, $\bar{\mathbf{Y}}_{-S_2}$ decreases with the size of S_2 as $|S_2| = n - |S_1|$. That is, if $T \subseteq S_1$ players are transferred to S_2, then $\bar{\mathbf{Y}}_{-S_2 \cup T} < \bar{\mathbf{Y}}_{-S_2}$. Hence,

$$z_1(p(.), \underset{S_1 \to S_2}{\bar{\Pi}^T}, \bar{\mathbf{Y}}_{-S_2 \cup T}) > z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2}) \text{ for any } T \subseteq S_1$$

Now suppose that $|T| = |S_1|$, then $z_1(p(.), \underset{S_1 \to S_2}{\bar{\Pi}^T}, \bar{\mathbf{Y}}_{-S_2 \cup S_1}) < |S_2 \cup S_1| = n$. On the other hand, if $X \subseteq S_2$ players are transferred to S_1 and $|X| = |S_2|$, then $z_1(p(.), \underset{S_1 \to S_2}{\bar{\Pi}^T}, \bar{\mathbf{Y}}_{-\emptyset}) = 0$. Hence, there must exist a fixed point such that $|S_2^*| = z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2^*})$ where $|S_2^*| \in (1, n-1)$.

Step 3: If the proposer in first round chooses $\sigma_i^P(\pi_0, N) = S_1$, where $|S_1| \in \{n - |S_2^*|, ...n-1\}$, and the responders agree $\sigma_j^R(\pi_0, N) = Y$ for all $j \in S_1, j \neq i$, then the equilibrium coalition structure is $\Pi^* = \{S_1, S_2\}$ where $|S_2| = n - |S_1|$.

Let $|S_2^*|$ represent a fixed point as discussed in step 2. Assume the initial proposer chooses $\sigma_i^P(\pi_0, N) = S_1$ where $|S_1| \geq n - |S_2^*|$ and the responders agree $\sigma_j^R(\pi_0, N) = Y$ for all $j \in S_1, j \neq i$. As $|S_2| = n - |S_1|$ and $z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2})$ is decreasing with $|S_2|$, we get

$$|S_2| \leq |S_2^*| < z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2^*}) \leq z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2}) \tag{37}$$

Therefore, by lemma 5, the optimal response of the active players in the second round is $\sigma_k^P(\pi_1, N) = S_2$ and $\sigma_\ell^R(\pi_0, N) = Y$ for all $\ell \in N \setminus S, \ell \neq k$ where $|S_2| = n - |S_1|$. Hence the equilibrium coalition structure is $\Pi^* = \{S_1, S_2\}$.

Step 4: To show that if the initial proposer chooses $\sigma_i^P(\pi_0, N) = S_1$ where $|S_1^*| \geq \max\{\frac{n}{2}, n - |S_2^*|\}$, then equilibrium coalition structure is $\Pi^* = \{S_1, S_2\}$ and the players' payoff is

$$\frac{p(S_1, \bar{\Pi}, \mathbf{Y})}{|S_1|} > \frac{1}{n} > \frac{p(S_2, \bar{\Pi}, \mathbf{Y})}{|S_2|}$$

Case A: $n - |S_2^*| \leq \frac{n}{2}$.

From (37), we get that

$$|S_2| < z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2}) \text{ for any } |S_2| \leq |S_2^*|.$$

As $|S_2^*| \geq \frac{n}{2}$, we conclude

$$|S_2| < z_1(p(.), \bar{\Pi}, \bar{\mathbf{Y}}_{-S_2}) \text{ for any } |S_2| \leq \frac{n}{2}.$$

Next, transfer $T \subseteq S_1$ to S_2 such that $|T| = \frac{|S_1|-|S_2|}{2}$. As $|S_2 \cup T| = |S_2| + |T| = \frac{|S_1|+|S_2|}{2} = \frac{n}{2} \leq |S_2^*|$, by Lemma 4 and symmetry we get

$$\frac{p\left(S_2, \bar{\Pi}, \bar{\mathbf{Y}}\right)}{|S_2|} < \frac{p\left(S_2 \cup T, \underset{S_1 \to S_2}{\bar{\Pi}^T}, \bar{\mathbf{Y}}\right)}{|S_2 \cup T|} = \frac{\frac{1}{2}}{\frac{n}{2}} = \frac{1}{n}.$$

From assumption 1A, we get

$$\left(\frac{p\left(S_1, \bar{\Pi}, \bar{\mathbf{Y}}\right)}{|S_1|}\right)\frac{|S_1|}{|S_1|+|S_2|} + \left(\frac{p\left(S_2, \bar{\Pi}, \bar{\mathbf{Y}}\right)}{|S_2|}\right)\frac{|S_2|}{|S_1|+|S_2|} = \frac{1}{|S_1|+|S_2|} = \frac{1}{n}. \tag{38}$$

$$\text{Therefore, if } \frac{p\left(S_2, \bar{\Pi}, \bar{\mathbf{Y}}\right)}{|S_2|} < \frac{1}{n}, \text{ then } \frac{p\left(S_1, \bar{\Pi}, \bar{\mathbf{Y}}\right)}{|S_1|} > \frac{1}{n}$$

Case B: $n - |S_2^*| > \frac{n}{2}$.

In this case, we show that for any $|S_2| < |S_2^*|$, the following inequality must always hold:

$$\frac{p\left(S_2, \bar{\Pi}, \bar{\mathbf{Y}}\right)}{|S_2|} < \frac{1}{n}. \tag{39}$$

We adopt the contra-positive approach for this purpose. Assume that

$$\frac{p\left(S_2, \bar{\Pi}, \bar{\mathbf{Y}}\right)}{|S_2|} \geq \frac{1}{n}. \tag{40}$$

Transfer $T \subseteq S_1$ to S_2 such that $|T| = \frac{|S_1|-|S_2|}{2}$. As $|S_2 \cup T| = |S_2| + |T| = \frac{|S_1|+|S_2|}{2} = \frac{n}{2} \leq |S_2^*|$, by symmetry we get

$$\frac{p\left(S_2 \cup \{T\}, \underset{S_1 \to S_2}{\bar{\Pi}^T}, \bar{\mathbf{Y}}\right)}{|S_2 \cup T|} = \frac{\frac{1}{2}}{\frac{n}{2}} = \frac{1}{n}.$$

This implies that for some transfer $0 < |T| < \frac{|S_1|-|S_2|}{2}$, the payoff to members of $S_2 \cup T$ has begun diminishing to the extent that at $|T| = \frac{|S_1|-|S_2|}{2}$ it is $\frac{1}{n}$ as shown above.

As the winning probability function $p(.)$ is well-behaved, this implies that the payoff to members of $S_2 \cup T$ will continue to diminish for all $|T| > \frac{|S_1|-|S_2|}{2}$. That is,

$$\frac{p\left(S_2 \cup T, \underset{S_1 \to S_2}{\bar{\Pi}^T}, \bar{\mathbf{Y}}\right)}{|S_2 \cup T|} < \frac{1}{n} \text{ for all } |T| > \frac{|S_1|-|S_2|}{2}.$$

However, this implication violates assumption 1A because

$$\frac{p\left(S_2 \cup \{T\}, \underset{S_1 \to S_2}{\bar{\Pi}^T}, \bar{\mathbf{Y}}\right)}{|S_2 \setminus \{T\}|} = \frac{1}{n} \text{ if } |T| = |S_1|.$$

In other words, if the grand coalition forms, then the payoff is $\frac{1}{n}$. Therefore, inequality (40) cannot hold.

Thus, as inequality (39) holds, by assumption 1A, we get

$$\frac{p\left(S_1, \bar{\Pi}, \bar{\mathbf{Y}}\right)}{|S_1|} > \frac{1}{n}$$

Step 5: Conclusion

By definition, a symmetric coalition structure $\Pi_K^s = \{C_1, C_2..., C_K\}$ where $|C_1| = |C_2|... = |C_K|$ and $K \geq 1$. The individual member payoff for any symmetric coalition structure is

$$\frac{p\left(C_i, \Pi_K^s, \bar{\mathbf{Y}}\right)}{|C_i|} = \frac{1}{n}$$

From step 4, we can conclude that if the initial proposer chooses $\sigma_i^P(\pi_0, N) = S_1$ where $|S_1| \geq \max\{\frac{n}{2}, n - |S_2^*|\}$, then equilibrium coalition structure is $\bar{\Pi} = \{S_1, S_2\}$ and the players' payoff is

$$\frac{p(S_1, \bar{\Pi}, \mathbf{Y})}{|S_1|} > \frac{1}{n}$$

Therefore, this strategy strictly dominates forming any symmetric coalition structure Π_K^s.

□

In the next result, we argue that if the number of active players is greater than its respective threshold, then the remaining players may split into further coalitions. The notation used below is described in at the beginning of Lemma 5. In this proof we show that if the number of active players is greater than its respective threshold, then the remaining players may split into further coalitions.

Lemma 6. *If $z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-A_t}\right) < A_t$, then the strategy profile that leads to the formation of the coalition structure $\Pi = \pi_t \cup \{C_{t+1}, C_{t+2}\}$ where $C_i \neq \emptyset$ for all $i \in \{t+1, t+2\}$ strictly dominates the formation of $\Pi = \pi_t \cup \{A_t\}$ where $A_t = C_{t+1} \cup C_{t+2}$ if there exists $C_{t+1} \subset A_t$ such that*
(A) $\frac{p\left(C_{t+1}, \Pi, \bar{\mathbf{Y}}\right)}{|C_{t+1}|} > \frac{p\left(A_t, \Pi', \bar{\mathbf{Y}}\right)}{|A_t|}$ where $\Pi = \pi_t \cup \{C_{t+1}, C_{t+2}\}$ and $\Pi' = \pi_t \cup \{A_t\}$.
(B) $z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-A_{t+1}}\right) \geq A_{t+1}$.

Proof. As $A_t > z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-A_t}\right)$ is the opposite of the condition states in Lemma 5, the remaining active set of players, A_t, could disintegrate to multiple coalitions. Condition (A) implies that there exists a sub-coalition $C_{t+1} \subset A_t$ that yields a higher payoff at round t given that the remaining players $A_t \setminus C_{t+1}$ form a coalition. Condition (B) ensures that. □

.

Proof of Corollary 1.1

Proof. From Lemma 6, the pre-condition for this result is $z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-A_t}\right) < A_t$ where A_t is the set of the remaining active players. As $z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-A_t}\right) \leq z_2\left(p(.), \Pi, \bar{\mathbf{Y}}_{-A_t}\right)$ by definition, a low $z_2\left(p(.), \Pi, \bar{\mathbf{Y}}_{-A_t}\right)$ would imply a low $z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-A_t}\right)$. This constitutes our statement (i) in Corollary 1.1.

If the marginal increase in winning probability $p(.)$ diminishes rapidly after the threshold $z_2(.)$, then the individual payoff from adding members to a coalition will tend to decrease. This would lead to condition (A) in Lemma 6. Hence, statement (ii) in Corollary 1.1.

Formation of three coalitions: At the beginning of the game, there are no coalitions formed. Hence, the coalition sub-structure at time $t = 0$ is $\pi_0 = \emptyset$. From Theorem 1, the initial proposer will propose a size $\sigma_i^P(\pi_0, N) = C_1$ where $|C_1| < n$. Hence, $f(\pi_0) = C_1$ and the coalition sub-structure at time $t = 1$ is $\pi_1 = \{C_1\}$. The set of active players at $t = 1$ is $A_1 = N \setminus C_1$.

Consider the bipartite coalition structure $\bar{\Pi} = \{C_1, A_1\}$. From step 2 of theorem 1, there exists a fixed point $X \subset N$ such that $A_1 > (resp. \leq) z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-A_1}\right)$ for all $|A_1| > (resp. \leq)\ |X|$. In what follows, we consider two cases: $|C_1| \geq n - |X|$ (or $A_1 \leq |X|$) and $|C_1| < n - |X|$ (or $A_1 > |X|$).

Case a: The initial proposer proposes a size $\sigma_i^P(\pi_0, N) = C_1$ where $|C_1| \geq n - |X|$.

From theorem 1, the remaining active players will form the second coalition; i.e. $f(\pi_1) = N \setminus C_1$. Hence, the initial proposer's payoff is

$$\frac{p\left(C_1, \Pi, \bar{\mathbf{Y}}\right)}{|C_1|} \quad \text{where} \quad |C_1| \geq n - x \quad \text{and} \quad \Pi = \{C_1, N \setminus C_1\}.$$

Let $\underline{C}^* \geq n - |X|$ be the size that maximises the payoff above.

Case b: The initial proposer proposes a size $\sigma_i^P(\pi_0, N) = C_1$ where $|C_1| < n - |X|$.

In this case, the remaining active players A_1 may break down into further coalitions depending on the nature of the contest success function $p(.)$. We discuss these conditions through the following sub-cases.

Sub-case b1: There exists $C_2 \subset A_1$ such that

$$\begin{array}{ll}(1) & A_2 \leq z_1\left(p(.), \Pi', \bar{\mathbf{Y}}_{-A_2\}}\right) \quad \text{where} \quad A_2 = A_1 \setminus C_2 \\ (2) & \dfrac{p\left(C_2, \Pi, \bar{\mathbf{Y}}\right)}{|C_2|} > \dfrac{p\left(A_1, \Pi', \bar{\mathbf{Y}}\right)}{|A_1|} \quad \text{where} \quad \Pi' = \{C_1, C_2, C_3\} \quad \text{and} \quad C_3 = N \setminus (C_1 \cup C_2).\end{array}$$

As $|C_1| < n - |X|$, it implies $A_1 < |X|$, and therefore $A_1 > z_1\left(p(.), \Pi, \bar{\mathbf{Y}}_{-A_1}\right)$. From Lemma 6 if the initial proposer proposes a size $\sigma_i^P(\pi_0, N) = C_1$ where $|C_1| < n - |X|$ and $\sigma_i^R(\pi_0, N) = Y$, then the formation of Π' strictly dominates Π. Note that exactly three coalitions form under these conditions. If Π' forms, then the initial proposer's payoff is

$$\frac{p\left(C_1, \Pi', \bar{\mathbf{Y}}\right)}{|C_1|}.$$

Let $\bar{C}^* < n - x$ be the size that maximises the payoff above for any three player coalitions that form.

Sub-case b2: There exists $C_2 \subset A_1$ such that

$$\begin{array}{ll}(1) & A_2 \gtreqless z_1\left(p(.), \Pi', \bar{\mathbf{Y}}_{-A_2\}}\right) \quad \text{where} \quad A_2 = A_1 \setminus C_2 \\ (2) & \dfrac{p\left(C_2, \Pi, \bar{\mathbf{Y}}\right)}{|C_2|} \leq \dfrac{p\left(A_1, \Pi', \bar{\mathbf{Y}}\right)}{|A_1|} \quad \text{where} \quad \Pi' = \{C_1, C_2, C_3\} \quad \text{and} \quad C_3 = N \setminus (C_1 \cup C_2).\end{array}$$

In this case, the formation of Π strictly dominates Π'. Thus, a bipartite coalition structure $\Pi = \{C_1, C_2\}$ emerges where $C_1 < n - x$. The initial proposer's payoff is

$$\frac{p\left(C_1, \Pi, \bar{\mathbf{Y}}\right)}{|C_1|} \quad \text{where} \quad |C_1| < n - x \quad \text{and} \quad \Pi = \{C_1, N \setminus C_1\}.$$

Let $\tilde{C}^* < n - |X|$ be the size that maximises the payoff above.

Sub-case b3: There exists $C_2 \subset A_1$ such that

$$\begin{array}{ll}(1) & A_2 > z_1\left(p(.), \Pi', \bar{\mathbf{Y}}_{-A_2\}}\right) \quad \text{where} \quad A_2 = A_1 \setminus C_2 \\ (2) & \dfrac{p\left(C_2, \Pi, \bar{\mathbf{Y}}\right)}{|C_2|} > \dfrac{p\left(A_1, \Pi', \bar{\mathbf{Y}}\right)}{|A_1|} \quad \text{where} \quad \Pi' = \{C_1, C_2, C_3\} \quad \text{and} \quad C_3 = N \setminus (C_1 \cup C_2).\end{array}$$

By reasoning similar to that we discussed till sub-case b1, this may lead to the formation of more than three coalitions in this case. The only difference is we need to develop cases a and b1 assuming the substructure $\pi_2 = C_1, C_2$ has formed instead of $\pi_1 = \{C_1\}$.

Formation of three coalitions:

To fix ideas, we state the conditions for the formation of exactly three coalitions. The equilibrium coalition structure Π^* consists of exactly three coalitions if there exists no $C_2 \subset A_1$ such that

$$\begin{array}{ll}(1) & A_2 > z_1\left(p(.), \Pi', \bar{\mathbf{Y}}_{-A_2\}}\right) \quad \text{where} \quad A_2 = A_1 \setminus C_2 \\ (2) & \dfrac{p\left(C_2, \Pi, \bar{\mathbf{Y}}\right)}{|C_2|} > \dfrac{p\left(A_1, \Pi', \bar{\mathbf{Y}}\right)}{|A_1|} \quad \text{where} \quad \Pi' = \{C_1, C_2, C_3\} \quad \text{and} \quad C_3 = N \setminus (C_1 \cup C_2).\end{array}$$

and

$$\frac{p\left(\bar{C},\Pi,\bar{\mathbf{Y}}\right)}{|\bar{C}|} > \max\left\{\frac{p\left(\underline{C},\Pi,\bar{\mathbf{Y}}\right)}{|\underline{C}|}, \frac{p\left(\tilde{C},\Pi,\bar{\mathbf{Y}}\right)}{|\tilde{C}|}\right\}$$

The equilibrium coalition structure is $\Pi^* = \{C_1^*, C_2^*, C_3^*\}$ where $|C_1^*| = \bar{C}$, $C_2^* = f(\{C_1\})$ and $C_3^* = N \setminus (C_1^* \cup C_2^*)$. As these conditions correspond to those of case b2, $f(\{C_1\}) \subset N \setminus C_1^*$ is the optimal coalition that forms at period $t = 2$. □

.

Proof of Proposition 1

Proof. Münster (2009) shows that if Assumption 1 is satisfied, then the winning probability function is represented by (6). The inflection point $z_1(.)$ for C_k's winning probability is obtained by computing the second order partial derivative of (6) with respect to Y_{C_k}.

$$z_1\left(p(.),\Pi,\mathbf{Y}_{-C_k}\right) = \begin{cases} \left(\frac{R_{C_k}-1}{R_{C_k}+1}\sum\limits_{C_j\neq C_k} Y_{C_j}^{R_{C_j}}\right)^{\frac{1}{R_{C_k}}} & \text{if } \; R_{C_k} > 1 \\ 0 & \text{if } \; R_{C_k} \leq 1 \end{cases}$$

Thus, (6) satisfies Assumption 2 if and only if $R_{C_k} > 1$. Similarly, the threshold $z_2(.)$ for C_k's winning probability is obtained by computing the cross partial derivative of (6).

$$z_2\left(p(.),\Pi,\mathbf{Y}_{-C_k}\right) = \left(\sum_{C_j\neq C_k} Y_{C_j}^{R_{C_j}}\right)^{\frac{1}{R_{C_j}}}.$$

Hence, (6) satisfies Assumption 3 as long as $R_{C_i} \geq 0$ for all $C_i \in \Pi$. Thus, this Assumption 3 characterises all classes of the generalised Tullock contest success functions.

Observe that these thresholds $z_1(.)$ and $z_2(.)$ stated above depend on the nature of the partition function determined by the impact factors, the coalition structure formed, and the vector of aggregate efforts by all rival coalitions. Further, the threshold is increasing with an increase in efforts by any rival coalition. All these observations are in agreement with the Assumptions 2 and 3. □

.

Proof of Proposition 2

Proof. We first explain the maximisation problem faced by any player $i \in N$. Next, we prove the result in two steps.

Maximisation Problem

Each player $i \in C_k$ maximizes (6) with respect to his effort y_i. The respective first order condition is given by

$$\frac{R_{|C_k|}}{|C_k|} \frac{\left(\sum_{C_j \neq C_k} Y_{C_j}^{R_{|C_j|}}\right)\left(y_i + \sum_{\ell \neq i} y_\ell\right)^{R_{|C_k|}-1}}{\left(\left(y_i + \sum_{\ell \neq i} y_\ell\right)^{R_{|C_k|}} + \sum_{C_j \neq C_k} Y_{C_j}^{R_{|C_j|}}\right)^2} - \alpha y_i^{\alpha-1} = 0 \text{ for all } i \in C_k. \tag{41}$$

As players are identical, FOC (41) would be symmetric for all members $i \in C_k$. Hence, the equilibrium effort would be $y_i^* = y_{C_k}^*$ for all $i \in C_k$ and $C_k \in \Pi$.

$$\frac{1}{y_i^*}\left(\frac{|C_k|^{R_{|C_k|}-2}\left(y_{C_k}^*\right)^{R_{|C_k|}} R_{|C_k|}\left(\sum_{C_j \neq C_k}\left(|C_j| y_{C_j}^*\right)^{R_{|C_j|}}\right)}{\left((|C_k| y_i^*)^{R_{|C_k|}} + X_{-C_k}^*\right)^2} - \alpha (y_i^*)^\alpha\right) = 0 \text{ for all } i \in C_k.$$

This implies that the aggregate effort by a coalition at equilibrium is $Y_{C_k}^* = |C_k| y_{C_k}^*$. Substituting in FOC (41), we obtain

$$f\left(Y_{C_k}^*\right) = \left(\frac{|C_k|}{Y_{C_k}^*}\right)^{\alpha-1}\left(g\left(Y_{C_k}^*\right) - \alpha\right) = 0 \text{ where } g\left(Y_{C_k}^*\right) = \frac{\left(Y_{C_k}^*\right)^{R_{|C_k|}-\alpha} R_{|C_k|} X_{-C_k}^*}{|C_k|^\alpha\left(\left(Y_{C_k}^*\right)^{R_{|C_k|}} + X_{-C_k}^*\right)^2}$$

Step 1: To show that there are exactly two roots of the function $f(.)$ if $\alpha > R_{|C_k|} + 1$.

For $R_{|C_k|} > 1$ and $\alpha > 1$, we get $f(Y_{C_k}^* = 0) = 0$. Hence, $Y_{C_k}^* = 0$ is a root of $f\left(Y_{C_k}^*\right)$. Next, if $\alpha > R_{|C_k|}$ then

$$g'\left(Y_{C_k}^*\right) = \frac{\left(Y_{C_k}^*\right)^{R_{|C_k|}-\alpha-1} R_{|C_k|} X_{-C_k}^*\left(X_{-C_k}^*\left(R_{|C_k|}-\alpha\right) - \left(Y_{C_k}^*\right)^{R_{|C_k|}}\left(R_{|C_k|}+\alpha\right)\right)}{|C_k|^\alpha\left(\left(Y_{C_k}^*\right)^{R_{|C_k|}} + X_{-C_k}^*\right)^3} < 0 \tag{42}$$

Hence, the function $g(.)$ strictly decreases with $Y_{C_k}^*$ if $\alpha > R_{|C_k|}$.

Further, if $\alpha > R_{|C_k|} + 1$ then

$$\lim_{Y_{C_k}^* \to 0} g\left(Y_{C_k}^*\right) = \infty \text{ and } \lim_{Y_{C_k}^* \to \infty} g\left(Y_{C_k}^*\right) = 0. \tag{43}$$

As $\alpha > 1$, by the intermediate value theorem, $g\left(Y_{C_k}^*\right)$ will intersect the horizontal line $y = \alpha$ exactly once at, say, $Y_{C_k}^* = K^* > 0$. Hence, if $\alpha > R_{|C_k|} + 1$ then there will be exactly two roots: $Y_{C_k}^* = 0$ and $Y_{C_k}^* = K^*$.

Step 2: To show that $Y_{C_k}^* = K^*$ maximises the payoff.

For $\alpha > R_{|C_k|} + 1$ we get that

$$\lim_{\epsilon \to 0} g(\epsilon) > 0 \tag{44}$$

Hence, $g(\epsilon) > g(0)$ and $Y^*_{C_k} = 0$ does not maximise the payoff. As the payoff function (6) is quasiconcave, the unique maximum will be $Y^*_{C_k} = K^* > 0$.

Therefore, the best-response of $Y^*_{C_k}\left(\sum_{C_j \neq C_k} Y_{C_j}\right) > 0$ for any non-zero aggregate rival's effort. Hence, we have an interior Nash equilibrium where $Y^*_{C_i} > 0$ for all $C_i \in \Pi$. □

.

Proof of Proposition 3:

Proof. In this proof, we show that the formation of an asymmetric bipartite coalition structure strictly dominates the formation of any symmetric coalition structure if the convexity of the cost function, measured by α, is sufficiently high.[12]

For any player $i \in C_k$, the FOC is given by

$$\frac{R}{|C_k|}\frac{\left(\sum_{C_j \neq C_k} Y^R_{C_j}\right)\left(y_i + \sum_{\ell \neq i} y_\ell\right)^{R-1}}{\left(\left(y_i + \sum_{\ell \neq i} y_\ell\right)^R + \sum_{C_j \neq C_k} Y^R_{C_j}\right)^2} - \alpha y_i^{\alpha-1} = 0 \quad \text{for all} \quad i \in C_k. \tag{45}$$

As players are identical, FOC (45) is symmetric for all members $i \in C_k$. Hence, the equilibrium effort is symmetric for all members of a coalition; i.e., $y^*_i = y^*_{C_k}$ for all $i \in C_k$ and $C_k \in \Pi$. From proposition 2, this effort will be strictly positive for all n players if $\alpha > R + 1$.

Lemma 7. *The grand coalition delivers the greatest individual payoff among all equal-sized coalition structures.*

Proof. Let $\Pi_K = \{C_1, C_2 ..., C_K\}$ denote any equal-sized coalition structure where $|C_k| = c_K = \frac{n}{K}$ for all $k = \{1, 2...K\}$. This equality reduces FOC (6) to

$$\frac{R\left(c_K y^*_{C_k}\right)^{R-1}(K-1)\left(c_K y^*_{C_k}\right)^R}{\left(K\left(c_K y^*_{C_k}\right)^R\right)^2} = \alpha(y^*_{C_k})^{\alpha-1}. \tag{46}$$

where $y^*_{C_k}$ is the equilibrium effort exerted by any individual player $i \in C_k$ given by

$$y^*_{C_k}) = \left(\frac{R(K-1)}{\alpha K n}\right)^{\frac{1}{\alpha}}. \tag{47}$$

Observe that $y^*_{C_k} > 0$ for any $K \geq 2$. As $K = 1$ implies the grand coalition, we get $y^*_{C_k} = 0$. To check the condition for positive effort, we compute the individual payoff at equilibrium

$$u^*(C_k, \Pi^K) = \frac{1}{n}\left(1 - \frac{R(K-1)}{\alpha K}\right) \tag{48}$$

[12]To jog the reader's memory, a symmetric coalition structure is one where the individual payoff to all n players is identical.

Hence, $y^*_{C_k} > 0$ if and only if

$$\alpha > \frac{R(K-1)}{K} \tag{49}$$

Observe that this condition matches that given in proposition 2. As the RHS of inequality (49) increases with K, players exert positive effort in any equal-sized coalition structure if

$$\alpha > \frac{R(n-1)}{n}$$

Conclusion: Among all equal-sized coalition structures, the grand coalition maximises individual payoff (48) that is obtained by setting $K = 1$:

$$u(N, \Pi^1) \;=\; \frac{1}{n}.$$

□

Lemma 8. *For any asymmetric bipartite coalition structure, the members of the larger coalition receive a payoff that greater than the grand coalition if and only if α is sufficiently high.*

Proof. This proof progresses in two steps.

Step 1: Equilibrium Analysis for Bipartite structures.

Symmetric bipartite structures: Let the bipartite coalition structure be denoted by $\Pi_{|C|} = \{C, D\}$ where $|D| = n - |C|$. If $\Pi_{|C|}$ is symmetric, i.e. $|C| = \frac{n}{2}$, then the equilibrium payoff is given by (48) where $K = 2$. Assuming condition (49) holds, this payoff will always be less than that obtained by forming the grand coalition.

Asymmetric bipartite structures Without loss of generality, let $|C| > \frac{n}{2}$. The individual payoff is given by (6). The FOC (45) reduces to

$$u'_i(C, \Pi_{|C|}, \mathbf{Y}) \;=\; \frac{R|C|^{R-2}|D|^R y_C^{R-1} y_D^R}{(|C|^R y_C^R + |D|^R y_D^R)^2} - \alpha y_C^{\alpha-1} = 0 \tag{50}$$

$$u'_j(D, \Pi_{|C|}, \mathbf{Y}) \;=\; \frac{R|C|^{R}|D|^{R-2} y_C^{R} y_D^{R-1}}{(|C|^R y_C^R + |D|^R y_D^R)^2} - \alpha y_D^{\alpha-1} = 0 \tag{51}$$

where $y_i = y_C$ for all $i \in C$ and $y_j = y_D$ for all $j \in D$. We obtain the following relation by solving the two FOCs above.

$$y_D \;=\; \left(\frac{|C|}{|D|}\right)^{\frac{2}{\alpha}} y_C \tag{52}$$

Substituting in (52) in FOC (50) we obtain the equilibrium individual efforts by the members of the two coalitions given by

$$y_C = \left[\frac{R\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}}{\alpha|C|^2\left(1+\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}\right)^2}\right]^{\frac{1}{\alpha}} \quad \text{and} \quad y_D = \left[\frac{R\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}}{\alpha\left(n-|C|\right)^2\left(1+\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}\right)^2}\right]^{\frac{1}{\alpha}}.$$

Similarly, the equilibrium payoff is given by

$$u_i(C,\Pi^b,\mathbf{Y}) = \frac{\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}}{|C|\left(1+\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}\right)}\left[1-\frac{R}{\alpha|C|\left(1+\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}\right)}\right]$$

Hence, members of coalition C and D exert positive efforts if and only if

$$\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)} > \max\left\{\frac{R}{\alpha|C|}-1, \frac{R}{\alpha(n-|C|)}-1\right\} = \frac{R}{\alpha(n-|C|)}-1$$

The term on the LHS of inequality above is increasing with the size $|C|$. Therefore, it would achieve a minimum at $|C|=\frac{n}{2}$. Hence, both coalitions would exert positive effort if

$$\alpha > \frac{R}{2(n-|C|)} \tag{53}$$

Again, this corroborates with the condition in proposition 2.

Step 2: To show that $u_i(C,\Pi^b,\mathbf{Y}) > \frac{1}{n}$ for all $\alpha > \bar{\alpha}$ where $\bar{\alpha} \in (2,\infty)$ is a constant.

We assume that (53) is satisfied, and therefore the equilibrium of efforts for Π^b is positive. Next, we show that $u_i(C,\Pi^b,\mathbf{Y})$ is increasing in α.

$$\frac{\partial u_i(C,\Pi^b,\mathbf{Y})}{\partial \alpha} = \frac{R\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}\left[\alpha+2\alpha|C|\left(1+\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}\right)\left(1-\frac{R}{\alpha|C|\left(1+\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}\right)}\right)\log\left(\frac{|C|}{n-|C|}\right)\right]}{\alpha^3|C|^2\left(1+\left(\frac{|C|}{n-|C|}\right)^{R\left(1-\frac{2}{\alpha}\right)}\right)^3} > 0 \tag{54}$$

As (53) is satisfied and $|C| > \frac{n}{2}$, the above expression is positive. Thus, $u_i(C,\Pi^b,\mathbf{Y})$ is increasing in α.

As we have established that $u_i(C,\Pi^b,\mathbf{Y})$ is increasing with coalition size $|C|$, we apply the intermediate value theorem to prove our result. For that purpose, we calculate the value of $u_i(C,\Pi^b,\mathbf{Y})$ at two limits

$$\lim_{\alpha\to 2} u_i(C,\Pi^b,\mathbf{Y}) = \frac{1}{2|C|}\left(1-\frac{R}{4|C|}\right) < \frac{1}{n} \quad \text{and} \quad \lim_{\alpha\to\infty} u_i(C,\Pi^b,\mathbf{Y}) = \frac{|C|^{R-1}}{(n-|C|)^R+|C|^R} > \frac{1}{n}.$$

By the intermediate value theorem, there must exist a constant $\bar{\alpha} \in (2,\infty)$ such that $u_i(C,\Pi^b,\mathbf{Y}) > \frac{1}{n}$ for all $\alpha > \bar{\alpha}$. □

If the size of the first coalition $|C| = (n-1)$, then its payoff is $u_i(C,\Pi^b,\mathbf{Y}) > \frac{1}{n}$ as shown in Step 2. Hence, the strategy where the first proposer chooses a coalition size $|C| = (n-1)$ and the second chooses $|D| = 1$ dominates forming the grand coalition.

Conclusion: If the first coalition is sufficiently large and if α is high, then forming a bipartite coalition structure strictly dominates forming the grand coalition.

□

Proof. In the table below, we write the winning probabilities and individual payoffs for every partition of N, where $n = 5$. For a representative coalition structure $\Pi = \{C_1, C_2..., C_K\}$, the first column indicates its coalition sizes their respective coalitions: $\{3,2,1\}$; and the second column indicates their respective winning probabilities.

Numeric coalition structure	Winning probabilities	Individual Payoff
$\{5\}$	(1)	$(\frac{1}{5})$
$\{4,1\}$	$(\frac{16}{17})(\frac{1}{17})$	$(\frac{4}{17})(\frac{1}{17})$
$\{3,2\}$	$(\frac{9}{2^{R_2}+9})(\frac{2^{R_2}}{2^{R_2}+9})$	$(\frac{3}{2^{R_2}+9})(\frac{2^{R_2-1}}{2^{R_2}+9})$
$\{3,1,1\}$	$(\frac{9}{11})(\frac{1}{11})(\frac{1}{11})$	$(\frac{3}{11})(\frac{1}{11})(\frac{1}{11})$
$\{2,2,1\}$	$(\frac{2^{R_2}}{2^{R_2+1}+1})(\frac{2^{R_2}}{2^{R_2+1}+1})(\frac{1}{2^{R_2+1}+1})$	$(\frac{2^{R_2-1}}{2^{R_2+1}+1})(\frac{2^{R_2-1}}{2^{R_2+1}+1})(\frac{1}{2^{R_2+1}+1})$
$\{2,1,1,1\}$	$(\frac{2^{R_2}}{2^{R_2}+3})(\frac{1}{2^{R_2}+3})(\frac{1}{2^{R_2}+3})(\frac{1}{2^{R_2}+3})$	$(\frac{2^{R_2-1}}{2^{R_2}+3})(\frac{1}{2^{R_2}+3})(\frac{1}{2^{R_2}+3})(\frac{1}{2^{R_2}+3})$
$\{1,1,1,1,1\}$	$(\frac{1}{5})(\frac{1}{5})(\frac{1}{5})(\frac{1}{5})(\frac{1}{5})$	$(\frac{1}{5})(\frac{1}{5})(\frac{1}{5})(\frac{1}{5})(\frac{1}{5})$

It is easy to check that $\{3,1,1\}$, $\{2,1,1,1\}$ and $\{1,1,1,1,1\}$ would not be the equilibrium as merging singleton coalitions generates efficiency gains. If $\pi_1 = \{1\}$, then the coalition structure is either $\{1,2,2\}$ or $\{1,4\}$. If $\pi_1 = \{2\}$, then the coalition structure is either $\{2,2,1\}$ or $\{2,3\}$. If $\pi_1 = \{3\}$, then the coalition structure is either $\{3,2\}$. If $\pi_1 = \{4\}$, then the coalition structure is either $\{4,1\}$. The equilibrium can be easily calculated using the payoffs in the above table depending on the value of R_2.

□

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