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A Game-Theoretic Implication of the Riemann Hypothesis

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Abstract. At the time of writing, the Riemann Hypothesis (RH) is one of the major unsolved problems in pure mathematics. In this note, a parameterized family of non-cooperative games is constructed with the property that, if RH holds true, then any game in this family admits a unique Nash equilibrium.

Keywords. Riemann hypothesis · Nash equilibrium

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1. Introduction

The Riemann Hypothesis (RH) is a famous open problem in the field of analytic number theory. The purpose of this note is to report on a curious game-theoretic implication of RH. Specifically, a parameterized family of games may be constructed with the property that if RH holds true, then each game in this family admits a unique Nash equilibrium.

There does not seem to exist prior academic work that connects RH to the theory of games. Nobel Laureate John Nash, whose contributions in the early 50s became the basis of modern game theory (Nash, 1950, 1951), and who had also solved Hilbert's 19th problem on partial differential equations, is understood to have worked on RH.[1] However, the bibliography of Milnor (1998) does not list any manuscript written by Nash with an obvious relationship to number theory.

The mathematical literature has come up with a large variety of conditions that are either necessary, sufficient, or equivalent to RH. In particular, Gröchenig (2020) related RH to the total positivity of a particular Fourier transform, and the observation made below draws heavily from his contribution.[2] However, the present analysis also crucially exploits novel game-theoretic arguments developed in fuller generality, in particular, by Ewerhart (2015, 2021).[3]

The sequel is structured as follows. Section 2 provides the necessary background on the Riemann zeta function and RH. A novel probability density function is introduced in Section 3. Section 4 presents the main result. Section 5 offers some discussion. Section 6 concludes. An Appendix contains supplementary proofs.

[1] According to a popular but unauthorized biography (Nasar, 1998), as well as to a Hollywood movie based upon it, Nash's presentation on the topic at Columbia University in 1959 became incomprehensible because of his beginning mental illness (see also Sabbagh, 2003).

[2] See also Katkova's (2007) related piece on totally positive sequences and work cited therein. Karlin's (1968) monograph is still the best introduction to the theory of total positivity.

[3] These methods have their origin in early work on two-person zero-sum games on the square (Karlin, 1957, 1959). Mattozzi and Levine (2021) usefully illustrated the validity of these general principles in all-pay contests with analytic payoffs and nonlinear costs. None of these earlier works dealt with the uniqueness of equilibrium in all-pay contests, however.

2. Background on the Riemann zeta function and the RH

The Riemann zeta function is an important special function in the field of number theory (Titchmarch, 1986; Davenport, 2013). For complex arguments $s \in \mathbb{C}$ with real part strictly exceeding one, it may be defined as the infinite sum

$$\zeta(s) = \sum\nolimits_{n=1}^{\infty} \frac{1}{n^s} \qquad (\operatorname{Re} s > 1). \tag{1}$$

This function admits a meromorphic extension[4] to the complex plane $\mathbb{C}$, being analytic except of a simple pole at $s = 1$. Moreover, with $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$ denoting the gamma function,

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) \tag{2}$$

is an entire function (i.e., it is analytic on $\mathbb{C}$) that is real-valued for real arguments and satisfies the functional equation

$$\xi(s) = \xi(1-s). \tag{3}$$

The Riemann zeta function vanishes at all even negative integers, i.e., at $s = -2, -4, \ldots$, and those zeros of ζ are called the trivial zeros. The Riemann hypothesis (RH), formulated by Riemann (1859), claims that all the non-trivial zeros of ζ lie on the *critical line* $\{\frac{1}{2} + it \,|\, t \in \mathbb{R}\}$. If true, the conjecture would admit powerful conclusions about the distribution of prime numbers.

Proving the RH is one of the seven problems for which the Clay Mathematics Institute awards a prize of one million dollars (Bombieri, 2000). Numerous interesting but ultimately partial results are available. For example, it is known that "more than 40 percent" of the nontrivial zeros of ζ lie on the critical line (Conrey, 1989). Moreover, starting with Turing (1953), substantial effort has been invested into attempts to reject RH using computational means. It has been shown, in particular, that the first 10^{13} non-trivial zeros lie exactly on the critical line (Gourdon,

[4] A meromorphic function is a complex-valued function in a complex variable that is analytic in all but possibly a discrete subset of its open domain, where all those singularities must be poles, i.e., vanish through multiplication with a suitably chosen polynomial.

2004).[5] However, as argued by Sarnak (2004, pp. 6-7), this need not mean that RH is "likely true." Finally, annoucements of alleged solutions to the problem are quite common (see, e.g., Schembri, 2018). At the time of writing, however, RH remains an open mathematical problem.

3. A probability density function

Consider the integral

$$f(t) = \frac{\xi(\frac{1}{2})}{2\pi} \int_{-\infty}^{+\infty} \frac{\cos(tx)dx}{\xi(\frac{1}{2}+x)} \qquad (-\infty < t < \infty). \tag{4}$$

Clearly, this functional form does not belong to the class of functions commonly employed in economics and statistical analysis (Johnson et al., 1995). Notwithstanding, provided that RH holds true, f is in fact a very well-behaved density function. This is suggested also by the numerically obtained graph of f, which is outlined in Figure 1.[6]

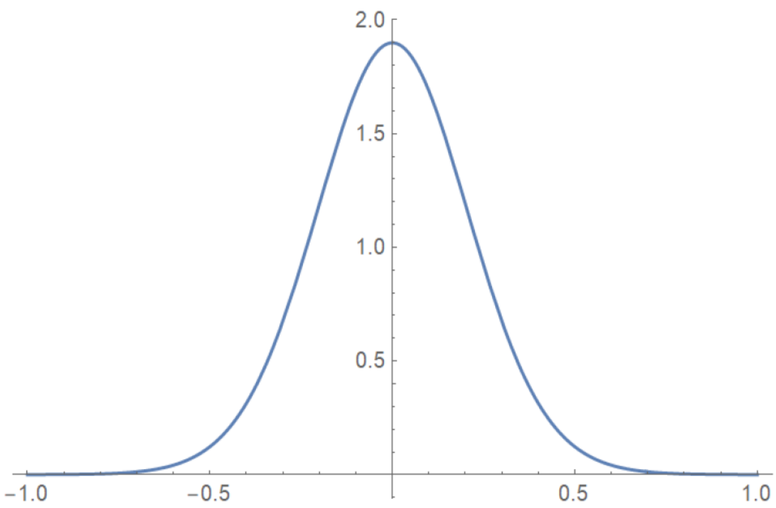

Figure 1. A density function.

[5] For details on the methods employed to ensure that all complex zeros of ζ up to a given height lie exactly on the critical line, see Edwards (1974, Ch. 8).

[6] All computations have been conducted using Wolfram's *Mathematica* 12.0.0 Kernel for Microsoft Windows (64-bit).

Lemma 1. *The integral (4) defines an analytic function f on $\mathbb{R}$ that is symmetric with respect to the origin. If RH holds true, then f is a proper Pólya frequency function, positive, logconcave, and exponentially diminishing with $\int_{-\infty}^{+\infty} f(t)dt = 1$.*

Proof.[7] See the Appendix. □

The property of f being a proper Pólya frequency function means that, in addition to f being integrable over $\mathbb{R}$, it is the case that, for any $n \geq 1$, and for any real parameters $a_1 > a_2 > \cdots > a_n$ and $b_1 > b_2 > \cdots > b_n$, the matrix

$$M_f \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} = \begin{pmatrix} f(a_1 - b_1) & f(a_1 - b_2) & \cdots & f(a_1 - b_n) \\ f(a_2 - b_1) & f(a_2 - b_2) & \cdots & f(a_2 - b_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(a_n - b_1) & f(a_n - b_2) & \cdots & f(a_n - b_n) \end{pmatrix} \tag{5}$$

has a positive determinant. As may be guessed, any normal density function is an example of a proper Pólya frequency function, and there are numerous other examples (Karlin, 1957, 1959; Ewerhart, 2021). What matters below, however, is that f might have this property.

4. Main result

Two players, player 1 and player 2, choose a nonnegative investment $x_1 \geq 0$ and $x_2 \geq 0$, respectively. Each player has to pay the chosen investment. There is a prize of common value $W > 0$.[8] Player 1 wins the prize with probability $p_1 = F(x_1 - x_2)$, where $F(t) = \int_{-\infty}^{t} f(\tau)d\tau$ is the cumulative distributions function associated with f. Player 2 wins with probability $p_2 = 1 - p_1$. Thus, payoffs are given by

$$\Pi_1(x_1, x_2) = F(x_1 - x_2)W - x_1, \tag{6}$$

$$\Pi_2(x_1, x_2) = F(x_2 - x_1)W - x_2. \tag{7}$$

[7] It follows from the proof that f has finite moments that all may be expressed in terms of the values of ξ and its derivatives at $s = \frac{1}{2}$. E.g., the mean is zero, and the variance is $\frac{\xi(\frac{1}{2})\xi''(\frac{1}{2}) - 2\xi'(\frac{1}{2})^2}{\xi(\frac{1}{2})^3}$. However, such observations are not needed in the sequel.

[8] The assumption of a common valuation is not crucial for the main observation and solely made to simplify the exposition.

This defines a non-cooperative game $G(W)$ for any $W > 0$.

As discussed in Ewerhart (2021), the Nash equilibrium of $G(W)$ is trivially in pure strategies as long as $W \leq 1/f(0)$, because in that case, players' payoff functions are strictly declining in their own strategy. Otherwise, i.e., if $W > 1/f(0)$, lack of quasiconcavity of the payoff function with respect to the own strategy makes it natural to consider randomized strategies, where the mixed extension is defined as usual by considering probability distributions on the Borel subsets of a suitably chosen compact interval (see, e.g., Dasgupta and Maskin, 1986). By Becker and Damianov (2006), $G(W)$ indeed admits a symmetric mixed-strategy Nash equilibrium strategy μ_1^*.

The following result provides a simple condition for equilibrium uniqueness.

Lemma 2. *Suppose that f is both analytic and a proper Pólya frequency function. Then, $G(W)$ admits precisely one Nash equilibrium, for any $W > 0$.*

Proof. See Ewerhart (2021, Thm. 1).[9] □

Combining Lemmas 1 and 2 yields the main result of the present paper.

Theorem 1. *If RH holds true, then $G(W)$ admits precisely one mixed-strategy Nash equilibrium, for any $W > 0$.*

Proof. Immediate from Lemmas 1 and 2 above. □

5. Discussion

It should be noted that it is not at all difficult to come up with a non-cooperative game that has precisely one Nash equilibrium if RH holds true. For example, the Prisoner's Dilemma admits a unique Nash equilibrium if RH holds true.[10]

[9] For the reader's convenience, a self-contained proof of Lemma 2 may be found in the Appendix.

[10] I am grateful to John Levy for providing this example.

As the conclusion is true, the implication holds regardless of the validity of the assumption. By the rules of Boolean logic.

The situation is different here. It is not known, and might never become known, if the conclusion of Theorem1 (equilibrium uniqueness in the two-player contest) is true or false. Thus, the conclusion is an open conjecture.[11] What Theorem 1 shows, therefore, is that one open conjecture (RH) implies another open conjecture. In particular, if the conclusion of equilibrium uniqueness could be shown to be wrong (which, as we know, is not feasible in the case of the Prisoner's dilemma), then the hypothesis would be proven wrong.[12]

G_1	L	R
T	θ,θ	0,0
B	0,0	-1,-1

G_2	L	R
T	1,θ	θ,0
B	0,1	0,0

Figure 2. The games G_1 and G_2.

It would nice, and certainly more satisfying, to find a game-theoretic conjecture that is logically equivalent to RH. In the abstract, this is actually not a big problem. E.g., one may even easily write down games for which the existence of a unique Nash equilibrium is equivalent to RH. To see this, consider the game G_1 depicted in Figure 2, where

$$\theta = \begin{cases} +1 & \text{if RH holds true} \\ -1 & \text{if RH does not hold true.} \end{cases} \tag{8}$$

If RH holds true, then G_1 admits (T,L) as the unique Nash equilibrium. If, however, RH does not hold true, then there are two Nash equilibria in pure strategies,

[11] Both RH and equilibrium uniqueness in the two-player contest may be characterized as being undecidable in the *practical* sense. Undecidability in the *logical* sense is a possibility here as well (i.e., RH and/or equilibrium uniqueness might be true but not provable), but this possibility is not crucial for the present discussion.

[12] A similar type of reasoning is used in the literature on the P versus NP problem in computational complexity theory (Cook, 1971), which it is a millennium problem like RH.

viz (T,R) and (B,L). Similarly, G_2 admits (T,L) as a unique pure-strategy Nash equilibrium if RH holds, and otherwise no pure-strategy Nash equilibrium.

In such examples, however, RH is used directly in the description of the payoff functions. That is, even if the strategy chosen by player 2 is correctly anticipated in G_1 or G_2, a human player 1 could not tell if T yields a higher payoff than B. In contrast, RH has no role in the definition of payoff *differences* in $G(W)$ between any two pure strategies, i.e., such payoff differences could, at least in principle, be approximated up to arbitrary accuracy without assuming RH. Indeed, if the opponent's pure strategy is correctly anticipated in $G(W)$, then any such payoff difference may be represented as

$$\Pi_1(x_1', x_2) - \Pi_1(x_1'', x_2) = -x_1' + x_1'' + W \int_{x_1''-x_2}^{x_1'-x_2} f(\tau) d\tau, \tag{9}$$

where the integral of the continuous function f over the compact interval converges regardless of whether RH holds true or not. In fact, given that best responses in the mixed extension of $G(W)$ have finite support, this argument extends to the relevant class of randomized strategies. For this reason, the two-player contest might be a more appealing example than G_1 or G_2, even though Theorem 1 does not capture a logical equivalence.

6. Concluding remarks

In a collected volume jointly edited by late John Nash and Michail Rassias, Connes (2016) offered a selective survey on RH. That paper actually used some game-theoretic terminology in the context of the Riemann-Roch theorem in tropical geometry and so-called chip-firing games on graphs. Notably, however, chip-firing games are one-player "solitaire" problems, which marks a difference to the theory outlined in the present paper.[13]

[13] The following characterization follows Baker and Norine (2007). Let (V, E) be a graph, with set of vertices V and set of edges E. Initially, each vertex $v \in V$ carries an integer number of

In a widely noticed contribution, Montgomery (1973) documents the observation that the distribution of distances between neighboring zeros of the Riemann zeta function on the critical line numerically matches the "repulsive" distribution of eigenvalues of certain random matrices known as the Gaussian Unitary Ensemble (GUE). Odlyzko (1987) added substantial numerical evidence to this observation. Rudnick and Sarnak (1994, 1996) formally established this correspondence in remarkable generality. While RH is not needed for their main result, adding RH as an assumption allows to strengthen their conclusions. More recently, Carmona et al. (2020) recovered the GUE distribution from limits of open-loop and closed-loop equilibria in certain N-player stochastic games as $N \to \infty$. However, as the authors remarked, the GUE naturally arises in numerous applied environments including, e.g., the spacing and arrival statistics of buses on a route in Cuernavaca, Mexico. Indeed, in contrast to the present inquiry, the identification of additional equilibria in their model would not invalidate RH.

Finally, in line with the celebrated Langlands program (Langlands, 1970), the RH has been extended to L-functions associated with automorphic forms on the general linear group GL_m (Sarnak, 2004). Conjecturally, this subsumes important historical predecessors such as Dedekind zeta functions, Artin L-functions, and Hasse-Weil zeta functions. In principle, the steps of the analysis above may be done analogously with these L-functions replacing ζ, which might allow a generalization of Theorem 1. However, additional assumptions may be needed, such as that the L-function does not vanish at $s = \frac{1}{2}$.[14] A more comprehensive investigation of such

chips. A vertex v which has a negative number of chips assigned to it is said to be "in debt." A move consists of a vertex v either borrowing (giving) one chip from (to) each of its neighbors. By a sequence of moves, the player strives to reach a configuration in which no vertex is in debt. A "winning strategy" is a sequence of moves that achieves such a configuration. Any initial assignment may be characterized as either admitting a winning strategy or not. If a winning strategy exists, one may also characterize winning strategies of minimal length.

[14] In the case of the Riemann zeta function, $\xi(\frac{1}{2}) > 0$ follows from the well-known product representation of Dirichlet's eta function, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$, where the sum on the left-hand side converges for $\mathrm{Re}(s) > 0$. Unfortunately, it seems hard to find a generalization of

questions is, however, beyond the scope of the present note.

Appendix. Technical proofs

Proof of Lemma 1. By the classical theory of the Riemann zeta function, $\xi > 0$ holds on the real line, with asymptotics $\ln \xi(s) \sim \frac{1}{2} s \ln s$ for $s \to \infty$ (Titchmarch, 1986, pp. 29-30). Therefore, as noted by Gröchenig (2020, p. 4), the Fourier transform of $x \mapsto 1/\xi(x + \frac{1}{2})$ exists, i.e., the integral

$$\widetilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(itx)dx}{\xi(\frac{1}{2} + x)} \qquad (-\infty < t < \infty) \tag{10}$$

is well-defined. Now, using Euler's formula

$$\exp(itx) = \cos(tx) + i \sin(tx), \tag{11}$$

and the functional equation (3), one observes that

$$\widetilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\cos(tx)}{\xi(\frac{1}{2} + x)} dx + \underbrace{\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\sin(tx)}{\xi(\frac{1}{2} + x)} dx}_{=0} \tag{12}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\cos(tx)}{\xi(\frac{1}{2} + x)} dx. \tag{13}$$

Thus, the integral in (4) is indeed well-defined. It is shown next that f is analytic. Take some $\varepsilon > 0$. For $z \in \mathbb{C}$ such that $|\mathrm{Im}(z)| < \varepsilon$, we have

$$\left| \frac{\exp(izx)}{\xi(\frac{1}{2} + x)} \right| \leq \frac{\exp(\varepsilon |x|)}{\xi(\frac{1}{2} + x)} \tag{14}$$

Moreover, from the beforementioned asymptotics of ξ on the real line,

$$\frac{\exp(\varepsilon |x|)}{\xi(\frac{1}{2} + x)} = \mathcal{O}\left(\exp\left(|x| \left(\varepsilon - \frac{\ln x}{2} \right) \right) \right). \tag{15}$$

Focusing on the case $|x| \geq \exp(2\varepsilon)$, one observes that the left-hand side of (15) is asymptotically diminishing at an exponential rate as $|x| \to \infty$, i.e.,

$$\frac{\exp(\varepsilon |x|)}{\xi(\frac{1}{2} + x)} = \mathcal{O}\left(\exp\left(-\varepsilon |x| \right) \right). \tag{16}$$

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this argument to more general L-functions. Cf., e.g., the discussion in Stark and Zagier (1980).

Thus, by Paley and Wiener (1934, Thm. I), $\widetilde{f}$ is analytic on $\mathbb{R}$.[15] Suppose that RH holds true. Then, as noted by Gröchenig (2020, Thm. 4), the equivalence between entire functions of the Pólya-Laguerre type and totally positive functions (Schoenberg, 1947, 1951) implies that the Fourier transform $\widetilde{f}$ is a Pólya frequency function. We have to strengthen this result somewhat. By Edwards (1974, Sec. 2.8), the "shifted" ξ-function admits the Hadamard product representation

$$\xi(s + \tfrac{1}{2}) = \xi(\tfrac{1}{2}) \prod_\rho \left(1 - \frac{s}{\rho - \frac{1}{2}}\right) e^{s/(\rho - \frac{1}{2})}, \tag{17}$$

where the product runs over the non-trivial zeros of the zeta function. Moreover, the infinite sum

$$\sum_\rho \frac{1}{|\rho - \frac{1}{2}|} \tag{18}$$

diverges. Hence, using Schoenberg and Whitney (1953, Thm. 1), $\widetilde{f}$ is even a proper Polya frequency function. In particular, $\widetilde{f}$ is globally positive by definition, and logconcave by Schoenberg (1951, Lemma 1). Next, it is claimed that $\widetilde{f}$ is diminishing at an exponential rate. One notes that

$$\widetilde{f}(0) = \int_{-\infty}^{+\infty} \frac{dx}{\xi(\frac{1}{2} + x)} > \int_{-\infty}^{+\infty} \frac{\cos(x)dx}{\xi(\frac{1}{2} + x)} = \widetilde{f}(1). \tag{19}$$

Therefore, $\ln \widetilde{f}(0) > \ln \widetilde{f}(1)$, so that the logconcavity of $\widetilde{f}$ implies that $\widetilde{f}$ is indeed tending to zero at an exponential rate. It remains to be shown that $\int_{-\infty}^{+\infty} f(t)dt = 1$. As $\widetilde{f}$ is positive and exponentially diminishing, it is absolutely integrable. Hence, from the Fourier inversion theorem (cf. Rudin, 1974, Thm. 9.11),

$$\int_{-\infty}^{+\infty} \widetilde{f}(t)dt = \frac{1}{\xi(\frac{1}{2})}. \tag{20}$$

Thus, $f = \xi(\frac{1}{2})\widetilde{f}$ is indeed a probability density function. This completes the proof of the lemma. □

[15] Alternatively, the analytic nature of $\widetilde{f}$ on the strip $|\mathrm{Im}(z)| < \varepsilon$ may be deduced directly from (16) using the conditions put forward by Mattner (2001).

Proof of Lemma 2.[16] Fix $W > 0$. As pointed out in the body of the paper, there is at least one (even symmetric) mixed-strategy Nash equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$ in $G(W)$. As F is analytic on the real line, so are expected payoffs against μ_1^*, considered as a function of x_2 (Ewerhart, 2015). Moreover, expected payoffs are positive for $x_2 = 0$, and negative for x_2 sufficiently large, hence not constant. Therefore, exploiting the general properties of analytic functions, there is a finite set S of pure strategies such that any pure best response is contained in S. Clearly, μ_2^* is a mixed best response to μ_1^*. Hence, the support of μ_2^* is finite and contained in S. Suppose there exists another equilibrium $\mu^{**} = (\mu_1^{**}, \mu_2^{**})$ in $G(W)$. By subsidizing each player with the effort of the opponent, the game $G(W)$ can be shown to be strategically equivalent to a two-person constant-sum game, which implies interchangeability. Therefore, μ_2^* is a best response also to μ_1^{**}. Consider the set of pure strategies $S' = \{y_1 > y_2 > \ldots > y_K\}$ in the support of either μ_1^* or μ_1^{**}.[17] It is claimed now that the system

$$M \begin{pmatrix} q_1 \\ \vdots \\ q_K \\ -\Pi/W \end{pmatrix} = \frac{1}{W} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ y_{K-1} \\ y_K \end{pmatrix} \in \mathbb{R}^{K+1}, \tag{21}$$

[16] Offered is merely a brief summary of the main arguments establishing Lemma 2. For additional details, the reader is referred to the original work by Ewerhart (2021, Thm. 1 and Lemma B.7).

[17] One can show that $y_K = 0$.

with the square matrix $M \in \mathbb{R}^{(K+1)\times(K+1)}$ defined by

$$M = \begin{pmatrix} \underbrace{f(y_1 - y_1)}_{=f(0)} & \cdots & f(y_1 - y_{K-1}) & f(y_1 - y_K) & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ f(y_{K-1} - y_1) & \cdots & \underbrace{f(y_{K-1} - y_{K-1})}_{=f(0)} & f(y_{K-1} - y_K) & 0 \\ F(y_{K-1} - y_1) & \cdots & \underbrace{F(y_{K-1} - y_{K-1})}_{=1/2} & F(y_{K-1} - y_K) & 1 \\ F(y_K - y_1) & \cdots & F(y_K - y_{K-1}) & \underbrace{F(y_K - y_K)}_{=1/2} & 1 \end{pmatrix}, \tag{22}$$

admits at most one solution. Indeed, subtracting the last row from the second-to-last row leads to

$$\det M = \begin{vmatrix} f(y_1 - y_1) & \cdots & f(y_1 - y_K) \\ \vdots & \ddots & \vdots \\ f(y_{K-1} - y_1) & \cdots & f(y_{K-1} - y_K) \\ F(y_{K-1} - y_1) - F(y_K - y_1) & \cdots & F(y_{K-1} - y_K) - \frac{1}{2} \end{vmatrix}. \tag{23}$$

Next, developing the determinant along the last row yields

$$\det M = \textstyle\sum_{k=1}^{K} \{(-1)^{k+K} (F(y_{K-1} - y_k) - F(y_K - y_k)) \tag{24}$$
$$\times \begin{vmatrix} f(y_1 - y_1) & \cdots & f(y_1 - y_{k-1}) & f(y_1 - y_{k+1}) & \cdots & f(y_1 - y_K) \\ \vdots & & \vdots & \vdots & & \vdots \\ f(y_{K-1} - y_1) & \cdots & f(y_{K-1} - y_{k-1}) & f(y_{K-1} - y_{k+1}) & \cdots & f(y_{K-1} - y_K) \end{vmatrix} \}.$$

Using

$$F(y_{K-1} - y_k) - F(y_K - y_k) = \int_{y_K}^{y_{K-1}} f(t - y_k)dt \qquad (k \in \{1, \ldots, K\}), \tag{25}$$

this becomes

$$\det M = \int_{y_K}^{y_{K-1}} \begin{vmatrix} f(y_1 - y_1) & \cdots & f(y_1 - y_K) \\ \vdots & \ddots & \vdots \\ f(y_{K-1} - y_1) & \cdots & f(y_{K-1} - y_K) \\ f(t - y_1) & \cdots & f(t - y_K) \end{vmatrix} dt. \tag{26}$$

As f is a proper Pólya frequency function, the determinant in (26) is seen to be positive for any $t \in (y_K, y_{K-1})$. Hence, $\det M > 0$. In particular, M is invertible, as claimed. It follows that there is at most one equilibrium. $\square$

References

Baker, M., Norine, S. (2007), Riemann-Roch and Abel-Jacobi theory on a finite graph, *archive.*

Becker, J.G., Damianov, D.S. (2006), On the existence of symmetric mixed strategy equilibria, *Economics Letters* **90**, 84-87.

Bombieri, E. (2000), *Problems of the Millenium: The Riemann Hypothesis*, Clay Mathematical Institute.

Carmona, R., Cerenzia, M., Palmer, A.Z. (2020), The Dyson and Coulomb Games, *Annales Henri Poincaré* **21**, 2897-2949.

Connes, A. (2016), "An Essay on the Riemann Hypothesis," in: Nash, J.F., Rassias, M.T. (eds.), *Open Problems in Mathematics*, Springer, New York.

Conrey, J.B. (1989), More than two fifths of the zeros of the Riemann zeta function are on the critical line, *Journal für die reine und angewandte Mathematik* **399**, 1-26.

Cook, S.A. (1971), The complexity of theorem-proving procedures, *Proceedings of the Third Annual ACM Symposium on Theory of Computing,* 151-158.

Dasgupta, P., Maskin, E. (1986), The existence of equilibrium in discontinuous economic games, I: Theory, *Review of Economic Studies* **53**, 1-26.

Davenport, H. (2013), *Multiplicative Number Theory (Vol. 74)*, 2nd ed., Revised by H.L. Montgomery, Springer.

Ewerhart, C. (2015), Mixed equilibria in Tullock contests, *Economic Theory* **60**, 59-71.

Ewerhart, C. (2021), A typology of military conflict based on the Hirshleifer contest, ECON Working Paper No. 400, University of Zurich. (under review)

Gourdon, X. (2004), The first zeros of the Riemann Zeta function, and zeros computation at very large height, *http://numbers.computation.free.fr*, last accessed: February 9, 2022.

Gröchenig, K. (2020), Schoenberg's Theory of Totally Positive Functions and the Riemann Zeta Function. *arXiv preprint* 2007.12889.

Johnson, N. L., Kotz, S., Balakrishnan, N. (1995), *Continuous Univariate Distributions*, Volume 2 (Vol. 289), John Wiley & Sons.

Katkova, O.M. (2007), Multiple positivity and the Riemann zeta-function, *Computational Methods of Function Theory* **7**, 13-31.

Karlin, S. (1957), On games described by bell shaped kernels, pp. 365-391 in: Dresher, M., Tucker, A.W., and Wolfe, P. (eds.), *Contributions to the Theory of Games III (Ann. Math. Studies* **39***)*, Princeton University Press.

Karlin, S. (1959), *Mathematical Methods and Theory in Games, Programming, and Economics*, Vol. II, Addison-Wesley.

Karlin, S. (1968), *Total Positivity*, Stanford University Press.

Langlands, R.P. (1970), Problems in the theory of automorphic forms. In: *Lectures in Modern Analysis and Applications III* (pp. 18-61). Springer, Berlin, Heidelberg.

Levine, D.K., Mattozzi, A. (2021), Success in contests, *Economic Theory*, published online, https://doi.org/10.1007/s00199-021-01367-4.

Mattner, L. (2001), Complex differentiation under the integral, *NAW* **5/2**, 32-35.

Milnor, J. (1998), John Nash and "A Beautiful Mind," *Notices of the American Mathematical Society*, November.

Nasar, S. (1998), *A Beautiful Mind: A Biography of John Forbes Nash Jr.*, Simon & Schuster.

Nash, J.F. (1950), Equilibrium points in n-person games, *Proceedings of the National Academy of Sciences* **36**, 48-49.

Nash, J.F. (1951), Non-cooperative games, *Annals of Mathematics* **54**, 286-295.

Odlyzko, A.M. (1987), On the distribution of spacings between zeros of the zeta function, *Mathematics of Computation* **48**, 273-308.

Paley, R., Wiener, N. (1934), *Fourier Transforms in the Complex Domain* (Vol. 19), American Mathematical Society.

Riemann, B. (1959), Über die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatshefte der Königlichen Preussischen Akademie der Wissenenschaften zu Berlin aus dem Jahre 1859* (1860), 671-680; also, *Gesammelte mathematische Werke und wissenschaftlicher Nachlass*, 2. Aufl. 1892, 145-155.

Rudin, W. (1974), *Real and Complex Analysis*, McGraw-Hill, New York.

Rudnick, Z., Sarnak, P. (1994), The n-level correlations of zeros of the zeta function, *Comptes Rendus de l'Académie des Sciences. Série 1, Mathématique,* **319**, 1027-1032.

Rudnick, Z., Sarnak, P. (1996), Zeros of principal L-functions and random matrix theory, *Duke Mathematical Journal* **81**, 269-322.

Sabbagh, K. (2003), *Dr. Riemann's Zeros*, Atlantic Books.

Sarnak, P. (2004), Problems of the Millennium: The Riemann Hypothesis, mimeo.

Schoenberg, I.J. (1947), On totally positive functions, Laplace integrals and entire functions of the Laguerre-Pólya-Schur type, *Proceedings of the National Academy of Sciences* **33**, 11-17.

Schoenberg, I.J. (1951), On Pólya frequency functions. I. The totally positive functions and their Laplace transforms, *Journal d'Analyse Mathématique* **1**, 331-374.

Schoenberg, I.J., Whitney, A. (1953), On Pólya frequence functions. III. The positivity of translation determinants with an application to the interpolation problem by spline curves, *Transactions of the American Mathematical Society* **74**, 246-259.

Stark, H.M., Zagier, D. (1980), A property of L-functions on the real line, *Journal of Number Theory* **12**, 49-52.

Titchmarsh, E.C. (1986), *The Theory of the Riemann Zeta-Function*, 2nd edition revised by D.R. Heath-Brown, Oxford University Press.

Turing, A.M. (1953), Some calculations of the Riemann zeta-function, *Proceedings of the London Mathematical Society* **3.1**, 99-117.