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The Lottery Contest is a Best-Response Potential Game

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Abstract. It is shown that the n-player lottery contest admits a best-response potential (Voorneveld, 2000, *Economics Letters*). This is true also when the contest technology reflects the possibility of a draw. The result implies, in particular, the existence of a non-trivial two-player zero-sum game that is best-response equivalent to a game with identical payoff functions.

Keywords. Potential games · Tullock contest · Best-response equivalence · Zero-sum games

JEL-Codes. C62, C72, D72

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1. Introduction

Potential games are interesting because they allow conclusions not only regarding existence and uniqueness of Nash equilibrium, but also regarding the outcome of dynamic and boundedly rational adjustment processes. Since Monderer and Shapley's (1996) seminal contribution, the literature has produced increasingly flexible variants of the initial concepts. One such generalization has led to the notion of a *best-response potential* (Voorneveld, 2000; Kukushkin, 2004; Dubey et al., 2006; Uno, 2007, 2011; Park, 2015). According to the definition, a game with continuous strategy spaces admits a best-response potential if there is a game with identical payoff functions that is *best-response equivalent* (henceforth *BR-equivalent*, etc.) to the original game, i.e., that has the same BR-correspondence, mapping any profile of pure strategies to a set of pure strategy profiles, as the original game.

This paper has two parts. In the first, we show that the n-player lottery contest admits a BR-potential. This holds true regardless of whether the contest allocates the prize with probability one (Haavelmo, 1954; Tullock, 1975; Bell et al., 1975; Baron, 1994) or there is a probability of a draw (Loury, 1979; Dasgupta and Nti, 1998; Blavatskyy, 2010; Jia, 2012).[1] In the second part of the paper, we exploit the strategic equivalence between contests and zero-sum games,[2] so as to derive potentially interesting implications of our result. In particular, it is shown that a non-trivial two-player zero-sum game may be BR-equivalent to a game with identical payoff functions.

The lottery contest and its natural generalizations have found widespread application in economics and political theory (Konrad, 2009). It corresponds to a Cournot game with isoelastic inverse demand and constant marginal costs. Deschamps (1975) proved convergence of fictitious play in a two-player Cournot oligopoly with strictly declining BR-functions. Thorlund-Peterson (1990) extended this result to an arbitrary number of firms. An exact potential exists for the Cournot game with linear demand (Slade, 1994). More generally, sufficient conditions for the existence of a BR-potential have been found for aggregative games that allow monotone BR-selections (Huang, 2002; Dubey et al., 2006; Jensen, 2010). However, all these methods do not apply to the lottery contest whose BR-function is not

[1] Like this paper, Dasgupta and Nti (1998) allow for both cases.

[2] By strategic equivalence, we mean vNM-equivalence (Morris and Ui, 2004), i.e., for each player, the payoff function in one game is equal to a positive constant times the payoff function in the other game, plus a term that depends only on the opponents' strategies. The strategic equivalence between contests and zero-sum games is implicit in Moulin and Vial (1978). Applications include Pavlov (2013), Ewerhart and Valkanova (2016), and Hwang and Rey-Bellet (2017).

monotone (Dixit, 1987). Also more recent examples of BR-potential games (Dragone et al., 2012; Bourlès et al., 2015) do not cover the case of the lottery contest.

2. The lottery contest admits a BR-potential

In the *lottery contest* G^a, with noise parameter $a \geq 0$, common valuation $V > 0$, and $n \geq 2$ players, each player $i \in \{1, ..., n\}$ simultaneously and independently chooses an effort $x_i \geq 0$, and subsequently receives a payoff of

$$u_i^a(x_1, ..., x_n) = \begin{cases} \dfrac{x_i}{a + \overline{x}} V - x_i & \text{if } a + \overline{x} > 0 \\ \\ V/n & \text{if } a + \overline{x} = 0, \end{cases} \tag{1}$$

where $\overline{x} = x_1 + ... + x_n$ denotes aggregate effort. The game G^a is known to possess a unique pure-strategy Nash equilibrium that is necessarily symmetric (Dasgupta and Nti, 1998).[3]

An n-person game $G = (X_1, ..., X_n, u_1, ..., u_n)$ with strategy spaces X_i and payoff functions $u_i : X \equiv X_1 \times ... \times X_n \to \mathbb{R}$ for players $i = 1, ..., n$ is called a *BR-potential game* (Voorneveld, 2000) if there exists a function $P : X \to \mathbb{R}$ such that

$$\arg\max_{x_i \in X_i} P(x_i, x_{-i}) = \arg\max_{x_i \in X_i} u_i(x_i, x_{-i}) \tag{2}$$

for any $i = 1, ..., n$ and any $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X_{-i} \equiv X_1 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_n$.

The following observation has, to the author's knowledge, not been documented in the literature.[4]

Proposition 1. *For any $a \geq 0$, the n-player lottery contest is a BR-potential game.*

Proof. Consider first the case $a > 0$. We claim that, in this case,

$$P^a(x_1, ..., x_n) = \left\{ a\overline{x} + \sum_{j<k} x_j x_k \right\} V - \frac{1}{3}(a + \overline{x})^3 \tag{3}$$

is a BR-potential for G^a. Indeed, differentiating (3), we find

$$\frac{\partial P^a(x_i, x_{-i})}{\partial x_i} = (a + \overline{x}_{-i})V - (a + \overline{x})^2, \tag{4}$$

[3] For $a = 0$, Dasgupta and Nti (1998) assume a different tie-breaking rule, viz. $u_i^0(0, ..., 0) = 0$ for $i = 1, ..., n$. The BR-correspondence is the same, however.

[4] Cf., e.g., Chowdhury and Sheremata (2015) and González-Sánchez and Hernández-Lerma (2016).

where $\overline{x}_{-i} = x_1 + ... + x_{i-1} + x_{i+1} + ... + x_n$. Moreover,

$$\frac{\partial^2 P^a(x_i, x_{-i})}{\partial x_i^2} = (-2) \cdot (a + \overline{x}) < 0, \tag{5}$$

i.e., the problem of maximizing $P^a(., x_{-i})$ subject to $x_i \geq 0$ is strictly concave. The unique solution $x_i^* \equiv x_i^*(x_{-i}, a)$ is given by

$$x_i^* = \begin{cases} \sqrt{(a + \overline{x}_{-i})V} - a - \overline{x}_{-i} & \text{if } \overline{x}_{-i} \leq V - a \\ \\ 0 & \text{if } \overline{x}_{-i} > V - a. \end{cases} \tag{6}$$

But this is just player i's best-response function in G^a. Hence,

$$\underset{x_i \geq 0}{\arg\max}\ P^a(x_i, x_{-i}) = \underset{x_i \geq 0}{\arg\max}\ u_i^a(x_i, x_{-i}), \tag{7}$$

as claimed.

Consider next the case $a = 0$. Denote by $\pi(x)$ the number of nonzero entries of the vector $x = (x_1, ..., x_n)$. We claim that

$$P^0(x_1, ..., x_n) = \begin{cases} (\sum_{j<k} x_j x_k)V - \frac{1}{3}\overline{x}^3 & \text{if } \pi(x_1, ..., x_n) \geq 2 \\ \\ -\frac{1}{3} x_j V^2 & \text{if } \pi(x_1, ..., x_n) = 1 \text{ and } x_j > 0 \\ \\ -\frac{1}{3}\frac{n-1}{n} V^3 & \text{if } \pi(x_1, ..., x_n) = 0 \end{cases} \tag{8}$$

is a BR-potential for G^0. To see this, suppose first that x_{-i} has at least two nonzero entries. Then, certainly $\pi(x_1, ..., x_n) \geq 2$, so that from (8),

$$P^0(x_1, ..., x_n) = (\sum_{j<k} x_j x_k)V - \frac{1}{3}\overline{x}^3. \tag{9}$$

Moreover, $u_i^0(., x_{-i})$ is differentiable, so that in straightforward extension of the case $a > 0$ considered above,

$$\underset{x_i \geq 0}{\arg\max}\ P^0(x_i, x_{-i}) = \underset{x_i \geq 0}{\arg\max}\ u_i^0(x_i, x_{-i}). \tag{10}$$

Suppose next that x_{-i} has precisely one nonzero entry $x_j > 0$. Then, $\pi(x_1, ..., x_n) = 2$ if $x_i > 0$, and $\pi(x_1, ..., x_n) = 1$ if $x_i = 0$. Hence, using (8) another time,

$$P^0(x_1, ..., x_n) = \begin{cases} x_i x_j V - \frac{1}{3}(x_i + x_j)^3 & \text{if } x_i > 0 \\ \\ -\frac{1}{3} x_j V^2 & \text{if } x_i = 0. \end{cases} \tag{11}$$

We have to show that the problem of maximizing $P^0(., x_{-i})$ subject to $x_i \geq 0$ has the unique solution that is given by player i's best-response function in G^0, i.e., by $x_i^* = \sqrt{x_j V} - x_j$

if $x_j \leq V$ and by $x_i^* = 0$ if $x_j > V$. From the above, it clearly suffices to verify that the problem $\max_{x_i \geq 0} P^0(x_i, x_{-i})$ has an interior solution if and only if $x_j < V$. But indeed, using (11), one can easily check that for $x_j > 0$,

$$\lim_{\substack{x_i \to 0 \\ x_i > 0}} P^0(x_i, x_{-i}) = -\frac{1}{3}x_j^3 > -\frac{1}{3}x_j V^2 = P^0(0, x_{-i}) \tag{12}$$

if and only if $x_j < V$, as illustrated in Figure 1.

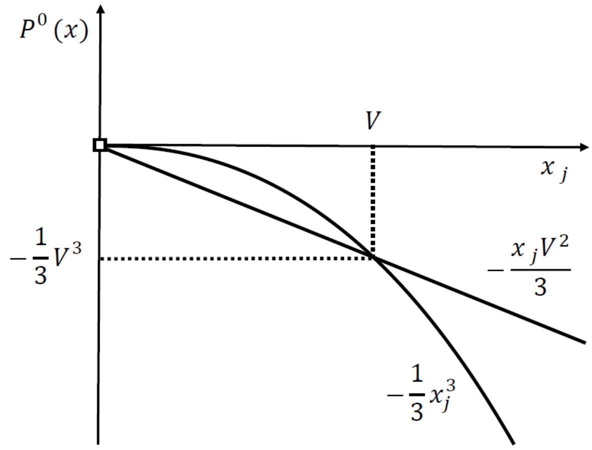

Figure 1. Constructing a BR-potential in the case $a = 0$.

Finally, suppose that all entries of x_{-i} are zero. Then, again from (8), $P^0(x_i, x_{-i}) = -\frac{1}{3}x_i V^2$ if $x_i > 0$, but $P^0(0, x_{-i}) < 0$, so that $\arg\max_{x_i \in X_i} P^0(x_i, x_{-i}) = \varnothing$. Similarly, $u_i^0(x_i, x_{-i}) = V - x_i$ if $x_i > 0$, but $u_i^0(0, x_{-i}) = V/n$, so that $\arg\max_{x_i \in X_i} u_i^0(x_i, x_{-i}) = \varnothing$. Thus, P^0 is indeed a BR-potential for G^0. □

3. Discussion

Note that G^a does not allow an exact potential for any $a \geq 0$. Indeed, for $i \neq j$,

$$\frac{\partial^2 u_i^a(x)}{\partial x_i \partial x_j} = \frac{(x_i - x_j) - (\sum_{k \neq i,j} x_k) - a}{(a + \overline{x})^3} V \tag{13}$$

is not symmetric with respect to i and j, as required by Monderer and Shapley's (1996) necessary condition. Along similar lines, it can be seen that the lottery contest is not a weighted potential game either.

For $a \geq V$, the function $u_i^a(\cdot, x_{-i})$ is strictly declining for any x_{-i} and i, and therefore, P^a is actually an ordinal potential for G^a. However, P^a is not even a generalized ordinal potential when $a < V$. Indeed, for $x_i = \varepsilon > 0$ small enough, $x_i' = V - a$, and $\overline{x}_{-i} = 0$,

$$u_i^a(x_i, x_{-i}) - u_i^a(x_i', x_{-i}) = \left(\frac{\varepsilon}{a+\varepsilon}V - \varepsilon\right) - \left(\frac{V-a}{a+(V-a)}V - (V-a)\right) > 0, \tag{14}$$

whereas, from (3) and (8),

$$P^a(x_i, x_{-i}) - P^a(x_i', x_{-i}) = \varepsilon\left\{a(V - a - \varepsilon) - \frac{\varepsilon^2}{3}\right\} - \frac{(V-a)^3}{3} < 0, \tag{15}$$

in conflict with the definition.[5]

Park (2015) argued that, if preferences are complete but the BR-set is empty, the BR-potential should generate the same preferences over strategies as the payoff function. To check this condition, note that for $x_{-i} = 0$,

$$P^0(x_i, x_{-i}) = \begin{cases} -\frac{1}{3}x_i V^2 & \text{if } x_i > 0 \\ -\frac{1}{3}\frac{n-1}{n}V^3 & \text{if } x_i = 0. \end{cases} \tag{16}$$

Thus, P^0 induces a preference for lower strategies among positive strategies $x_i > 0$, and an indifference between $x_i = 0$ and $x_i = \frac{n-1}{n}V$. This is likewise true for preferences reflecting player i's payoffs when $x_{-i} = 0$,

$$u_i^0(x_i, x_{-i}) = \begin{cases} V - x_i & \text{if } x_i > 0 \\ V/n & \text{if } x_i = 0. \end{cases} \tag{17}$$

Thus, P^0 satisfies Park's condition.

Proposition 1 extends to any convex combination between a purely random allocation and the lottery contest (Haavelmo, 1954; Baron, 1994). The same is true for Amegashie's (2006) contest with minimum efforts.

4. Implications

This section presents two implications of Proposition 1.

An n-player game G will be called $n-1$ *multilateral* if for any $i = 1, ..., n$, there exist functions $h_{ij} : X_{-j} \to \mathbb{R}$ for $j \in \{1, ..., n\}$, such that $u_i(x) = \sum_{j=1}^n h_{ij}(x_{-j})$. Any zero-sum equivalent potential n-player game is necessarily equivalent to an $n-1$ multilateral game (Brânzei et al., 2003; Hwang and Rey-Bellet, 2017). As the following result shows, this need not be so for a zero-sum equivalent BR-potential game.

[5] This leaves, of course, the theoretical possibility that another function might be a (generalized) ordinal potential for G^a when $a < V$.

Proposition 2. *For any $n \geq 2$, there exists a zero-sum equivalent BR-potential n-player game that is not equivalent to an $n-1$ multilateral game.*

Proof. Recall that G^0 is strategically a zero-sum game.[6] To see that G^0 is not $n-1$ multilateral, note that

$$\frac{\partial^{n-1} u_i^a(x_i, x_{-i})}{\partial x_1 ... \partial x_{i-1} \partial x_{i+1} ... \partial x_n} = (-1)^{n-1}(n-1)! \frac{x_i}{(x_i + \overline{x}_{-i})^n}, \tag{18}$$

depends nontrivially on x_i, which is impossible for an $n-1$ multilateral game. □

Call a two-player game *trivial* if each player's BR-correspondence is constant. For $n = 2$, the statement of Proposition 2 may be strengthened as follows.

Corollary 1. *There exists a non-trivial two-person zero-sum equivalent BR-potential game.*

Hwang and Rey-Bellet (2017) have shown that, for $n \geq 3$, the n-player Cournot game with linear demand, which admits an exact potential, is zero-sum equivalent. This is not true for $n = 2$, however, so that Corollary 1 may indeed be of interest.

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[6] For a direct proof, consider the n-player game G with payoffs $u_i(x) = u_i^0(x) - \frac{V}{n} + x_{i+1}$, where $x_{n+1} = x_1$. Since G is zero-sum, this proves the claim.

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