Lecture 8: Consumption-Savings Decisions

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1 Plan

1. A little historical background on models of consumption
2. Some evidence (!)
3. Studying the incomplete markets consumption model (which we saw last class as an example) with recursive tools

2 The Permanent Income Hypothesis

- Recall the consumption-savings problem under certainty

  \[
  \max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t)
  \]

  s.t.

  \[a_{t+1} = Ra_t + y_t - c_t\]

  and a No Ponzi condition

- Assuming \( R \) is constant, the NPV budget constraint is

  \[\sum_{t=0}^{\infty} R^{-t} c_t = \sum_{t=0}^{\infty} R^{-t} y_t\]

- Euler equation

  \[u'(c_t) = \beta Ru'(c_{t+1})\]

- Special case: \( R\beta = 1 \)
- Euler equation implies constant consumption
- Budget

\[ \frac{1}{1 - R^{-1}} c = \sum_{t=0}^{\infty} R^{-t} y_t \]

\[ \Rightarrow c = \frac{R - 1}{R} \sum_{t=0}^{\infty} R^{-t} y_t \]

- E.g. if \( y_t = y \) constant, consume income every period
- For later use, it is helpful to rewrite this as (exercise! hint: use the sequential budget constraint)

\[ c = \frac{R - 1}{R} \left[ Ra_t + \sum_{s=0}^{\infty} y_{t+s} R^{-s} \right] = \frac{R - 1}{R} \left[ Ra_t + y_t + \sum_{s=1}^{\infty} y_{t+s} R^{-s} \right], \]

which holds for any period \( t \)
- Now assume that:
  - the income process is uncertain
  - the household acts as though it wasn’t: certainty equivalent behavior (CEQ)
- Certainty equivalent behavior is not optimal in general
- But it’s optimal if preferences are quadratic - we’ll see this later (idea: linear Euler equations)
- Consumption becomes

\[ c_t = \frac{R - 1}{R} \left[ Ra_t + y_t + E_t \left( \sum_{s=1}^{\infty} R^{-s} y_{t+s} \right) \right] \] (1)

- Idea: each period, you recompute the expected NPV of future income, forget it’s uncertain and recalculate the optimal consumption path
- Set current consumption equal to annuity value of assets and expected future income
- This is called the permanent income hypothesis (PIH): given “permanent” income

\[ y_t^p = y_t + E_t \left( \sum_{s=1}^{\infty} R^{-s} y_{t+s} \right), \]
\( c_t \) should be a function of \( y_t^p \) and not independently of current income \( y_t \) (Milton Friedman)

- Alternatively, we can again rewrite (1) using the sequential budget constraint as

\[
\frac{c_t}{R - 1} = \left( E_t \left[ \sum_{s=0}^{\infty} R^{-s} y_s \right] \right)
\]

- Notice timing

### 2.1 Contrast with traditional Keynesian view

- Keynes (1936):

  “The fundamental psychological law, upon which we are entitled to depend with great confidence both a priori from our knowledge of human nature and from the detailed facts of experience, is that men are disposed, as a rule and on the average, to increase their consumption as their income increases but not by as much as the increase in the income.”

- Keynesian consumption function:

\[
\frac{c_t}{y_t} = \alpha_0 + \alpha_1 y_t
\]

- Keynesian marginal propensity to consume (MPC): \( \alpha_1 < 1 \)

- PIH MPC

  - Marginal propensity to consume out of wealth: \( \frac{\partial c_t}{\partial (E_t | \sum_{s=0}^{\infty} R^{-s} y_s)} = \frac{R-1}{R} \approx 0.04. \) small!

    - same for MPC out of assets \( a_t \).

  - Marginal propensity to consume out of a purely temporary income increase: \( \frac{\partial c_t}{\partial y_t} = \frac{R-1}{R} \) (holding \( E_t [\sum_{s=1}^{\infty} R^{-s} y_{t+s}] \) constant)

  - Marginal propensity to consume out of a permanent income increase \( \frac{\partial c_t}{\partial \bar{y}} = 1 \)

  - In general, MPC out of shocks to current income depends on income process. It could be greater than 1 if income growth is highly serially correlated. See below.
2.2 The random walk prediction

- Take first difference of (1) and use sequential budget constraint (exercise!):

\[
\Delta c_t = c_t - c_{t-1} = \frac{R-1}{R} \left( \sum_{s=0}^{\infty} R^{-s} (E_{t}y_{t+s} - E_{t-1}y_{t+s}) \right)
\] (2)

- Change in consumption comes from revisions in permanent income (i.e. news arriving at \( t \) about the expected present value of future income)

- Therefore

\[ E_{t-1}(\Delta c_t) = 0 \]

- Intuition: consumer has no access to insurance, so consumption smoothing implies minimizing \( \Delta c \)

- Testable!

1. Regress \( \Delta c_t \) on information known at \( t-1 \): should be unpredictable!

2. Relax \( \beta R = 1 \) but keep the certainty-equivalence assumption:

\[ c_t = a_0 + a_1 c_{t-1} + \varepsilon_t \]

Regress \( c_t \) on \( c_{t-1} \) and information known at \( t-1 \). This is like the previous case but you allow \( a_0 \neq 0 \) and \( a_1 \neq 1 \).

3. Relax certainty-equivalence

- Test Euler equation:

\[ E_{t-1} \left( \beta R \frac{u'(c_t)}{u'(c_{t-1})} \right) = 1 \]

- CRRA case with riskless interest rate

\[
E_{t-1} \left( \beta R \left( c_t \over c_{t-1} \right)^{-\sigma} \right) = 1
\]

\[ c_t^{-\sigma} = c_{t-1}^{-\sigma} (\beta R)^{-1} + \varepsilon_t \]

- Assume values for \( \sigma \)

- Regress \( c_t^{-\sigma} \) on \( c_{t-1}^{-\sigma} \) and information known at \( t-1 \)
• Evidence from Hall [1978]:

<table>
<thead>
<tr>
<th>Equation No. and Equation</th>
<th>$R^2$</th>
<th>s</th>
<th>D-W</th>
<th>F</th>
<th>F*</th>
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<tbody>
<tr>
<td>3.1 $c_t = -16 + 1.024 c_{t-1} - .010 y_{t-1}$</td>
<td>.9988</td>
<td>14.7</td>
<td>1.71</td>
<td>.1</td>
<td>3.9</td>
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<td>(11) (.044) (.032)</td>
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<tr>
<td>3.2 $c_t = -23 + 1.076 c_{t-1} + .049 y_{t-1} - .051 y_{t-2}$</td>
<td>.9989</td>
<td>14.4</td>
<td>2.02</td>
<td>2.0</td>
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<td>(11) (.047) (.043) (.032)</td>
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<td></td>
<td>$-.023 y_{t-3} - .024 y_{t-4}$</td>
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<td>(1.051) (.037)</td>
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<tr>
<td>3.3 $c_t = -25 + 1.115 c_{t-1} + \sum_{t=1}^{4} \beta y_{t-s} - .077 \sum \beta_i = .077$</td>
<td>.9988</td>
<td>14.6</td>
<td>1.92</td>
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<td>(11) (.054)</td>
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• If previous consumption was based on all information consumers had at the time, past income should not contain any additional explanatory power about current consumption above past consumption. Supported in the data.

• As long as income is somewhat predictable, this distinguishes PIH from Keynesian or more generally from:

$$c_t = D(L)y_t$$

2.3 Test of consumption smoothing

• Start from (2), estimate stochastic process for income and check.

• Example: $y_t$ follows an MA(1) process

$$y_t = \varepsilon_t + \beta \varepsilon_{t-1}$$

• Using (2),

$$\Delta c_t = \frac{R-1}{R} \left( \sum_{s=0}^{\infty} R^{-s} (E_t y_{t+s} - E_{t-1} y_{t+s}) \right)$$

$$= \frac{R-1}{R} \left( y_t - E_{t-1} y_t + R^{-1} (E_t y_{t+1} - E_{t-1} y_{t+1}) \right)$$

$$= \frac{R-1}{R} \left( 1 + R^{-1} \beta \right) \varepsilon_t$$

since $E_t y_{t+1} = \beta \varepsilon_t$ and $E_{t-1} y_{t+s} = 0$ for all $s \geq 1$
• Hence this is an example where the stochastic process for income features serial correlation (if $\beta > 0$) and

$$\frac{\partial c_t}{\partial \epsilon_t} > \frac{R - 1}{R}$$

• MPC from innovation to current income depends on persistence of income process if we do not control for permanent income.

• Could compute this for more general ARMA processes for income

$$\alpha(L)y_t = \beta(L)\epsilon_t$$

• Campbell and Deaton [1989]: estimate

$$\Delta y_t = 8.2 + 0.442\Delta y_{t-1} + \epsilon_t \quad \text{with } \sigma_\epsilon = 25.2$$

• Income growth positively correlated $\Rightarrow$ permanent income more volatile than current income.

• Using (2):

$$\Delta c_t = \frac{R}{0.558 + R - 1}\epsilon_t = 1.78\epsilon_t$$

for $R = 1.01$ quarterly.

• Standard deviation of $\Delta c_t$ should be 1.78 times $\sigma_\epsilon$.

• Data: 27.3 for total; 12.4 for nondurables, less than $\sigma_\epsilon = 25.2$: “excess smoothness”!

• One explanation: slow adjustment to innovations in permanent income

2.4 Test of Euler equation (without assuming values for $\sigma$)

• Derive log-linear version of Euler equation [Hansen and Singleton, 1983]:

$$\mathbb{E}_t \left( \beta R_{t+1} \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \right) = 1$$
• Define
\[ z_{t+1} \equiv R_{t+1} \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \]

• Assume \( z_{t+1} \) is lognormal given time-\( t \) information:
\[ \log z_{t+1} = \log R_{t+1} - \sigma \Delta \log c_{t+1} \sim N(\mu_t, v_t) \]

• Define
\[ \epsilon_{t+1} \equiv \log z_{t+1} - \mu_t \]
and note
\[ \epsilon_{t+1} \sim N(0, v_t) \]

• Using the properties of the lognormal distribution:
\[ \mathbb{E}_t (z_{t+1}) = \exp \left[ \mu_t + \frac{v_t}{2} \right] \]

• Now use the Euler equation:
\[ \mathbb{E}_t (z_{t+1}) = \frac{1}{\beta} \]
\[ \exp \left[ \mu_t + \frac{v_t}{2} \right] = \frac{1}{\beta} \]
\[ \mu_t + \frac{v_t}{2} = -\log \beta \]
\[ \log (z_{t+1}) - \epsilon_{t+1} + \frac{v_t}{2} = -\log \beta \]
\[ \log R_{t+1} - \sigma \Delta \log c_{t+1} - \epsilon_{t+1} + \frac{v_t}{2} = -\log \beta \]
\[ \Delta \log c_{t+1} = \left[ \frac{\log \beta}{\sigma} + \frac{v_t}{2\sigma} \right] - \frac{\log R_{t+1}}{\sigma} - \frac{\epsilon_{t+1}}{\sigma} \quad (3) \]

• \( \Delta \log c_{t+1} \) should not be predictable with time-\( t \) information other than interest rates.

• Campbell and Mankiw [1990] estimate:
\[ \Delta \log c_{t+1} = \mu + \lambda \Delta \log y_{t+1} + \theta r_{t+1} + \epsilon_{t+1} \]
using lagged values of consumption or income growth or interest rates as instruments for \( \Delta \log y_{t+1} \).
"Excess sensitivity" to predictable current income

- Campbell and Mankiw [1990]: a fraction of consumers just consume their income. Liquidity constraints?

- Campbell and Deaton [1989]: delayed response of consumption to innovations. Would also explain "excess smoothness"

- Statistically: predictable changes in consumption

- Why the difference with Hall [1978]? 1950:1 (large payments to WWII veterans)?

- Tests generally rely on aggregate data. Aggregation issues:
  - across goods
  - across agents: Euler equations non-linear (Attanasio and Weber)
  - across time: data averaged over continuous time, which introduces serial correlation

- Carroll [1997]: don’t ignore the $v_t$ term in (3)!

- "Buffer-stock" behavior predicts systematic relation between $v_t$ and wealth/income ratio

- More generally: “precautionary savings”

3 Precautionary savings: two-period example

- No complete markets

- Only riskless borrowing-saving
• Uncertain future income

\[
\max_{a} u (y_0 - a) + \beta \sum_{s} \Pr(s) u (Ra + y_1(s))
\]

• FOC:

\[
-u'(y_0 - a^*) + \beta R \sum_{s} \Pr(s) u'(Ra^* + y_1(s)) = 0
\]

(4)

• Define \(\hat{a}\) as the solution to

\[
-u'(y_0 - \hat{a}) + \beta Ru'(R\hat{a} + \sum_{s} \Pr(s) y_1(s)) = 0
\]

• Is it the case that \(a^* = \hat{a}\) (certainty equivalence)?

**Proposition 1.** \(a^* > \hat{a}\) if \(u'(\cdot)\) is convex

*Proof.* Assume the contrary:

\[
a^* \leq \hat{a}
\]

\[
\Rightarrow u'(Ra^* + y_1(s)) \geq u'(R\hat{a} + y_1(s)) \quad \forall s
\]

\[
\sum_{s} \Pr(s) u'(Ra^* + y_1(s)) \geq \sum_{s} \Pr(s) u'(R\hat{a} + y_1(s))
\]

\[
> u'\left(R\hat{a} + \sum_{s} \Pr(s) y_1(s)\right) \quad (u'\text{ convex})
\]

\[
= \frac{u'(y_0 - \hat{a})}{\beta R} \quad \text{by definition}
\]

\[
\geq \frac{u'(y_0 - a^*)}{\beta R} \quad (a^* \geq \hat{a})
\]

which contradicts (4)

• “Precautionary savings” if \(u'''(\cdot) > 0\)

• Inevitable as \(c \to \infty\)
• True for CRRA:

\[ u(c) = \frac{c^{1-\sigma}}{1-\sigma} \]
\[ u'(c) = c^{-\sigma} \]
\[ u''(c) = -\sigma c^{-\sigma-1} \]
\[ u'''(c) = \sigma (1 + \sigma) c^{-\sigma-2} > 0 \]

• For quadratic preferences: certainty equivalence


• We talked about this as one example of a dynamic optimization problem under uncertainty. Now we’ll analyze it in detail

• Recursive representation:

\[
V(x, s) = \max_{c, x'(s')} u(c) + \beta \sum_{s'} \Pr(s'|s) V(x'(s'), s') \tag{5}
\]
\[ s.t. \quad x'(s') \leq R[x - c] + y(s') \]
\[ c \leq x + b \]

\[
u'(c) - \sum_s \lambda(s) R - \mu = 0
\]
\[
\beta \Pr(s'|s) \frac{\partial V(x'(s'), s')}{\partial x} - \lambda(s) = 0
\]
\[ \Rightarrow u'(c) \geq \beta R \sum_{s'} \Pr(s'|s) \frac{\partial V(x'(s'), s')}{\partial x} \]

with equality if \( c < x + b \)

• iid shocks and borrowing constraint not binding:

\[ u'(c) = \beta \mathbb{E}[V'(x')] \]

again: precautionary savings if \( V' \) convex, but \( V''' \) is now endogenous!
• result: $u''' > 0 \Rightarrow V''' > 0$ (Sibley 1975)

• Envelope condition
  \[
  \frac{\partial V(x,s)}{\partial x} = u'(c(x,s))
  \]

• Euler equation:
  \[
  u'(c) \geq \beta R \sum_{s'} \Pr(s'|s) u'(c(x'(s'), s'))
  \]
  with equality if $c < x + b$

4.1 The $\beta R = 1$ case

• For $\beta R = 1$, the Euler equation is
  \[
  u'(c) \geq \sum_{s'} \Pr(s'|s) u'(c(x'(s'), s'))
  \]
  so marginal utility follows a supermartingale

**Theorem 1** (Doob). Let \{\{Z_t\}\} be a nonnegative supermartingale. Then $Z_t \to_{a.s.} Z$, where $Z$ is a random variable with $\mathbb{E}[Z] < +\infty$

• What does it mean for a sequence of random variables to converge a.s. to a random variable?

  1. The sequence converges
  2. The point towards which it converges is random, i.e. could be different for different realizations of the sequence

**Example 1.** Suppose

\[
Z_t = \begin{cases}
Z_{t-1} - u_t & \text{if } t \leq 7 \\
Z_{t-1} & \text{otherwise}
\end{cases}
\]

where $u_t \sim U[0,1]$, and $Z_0 = 8$. Then $Z_t$ is a nonnegative supermartingale. It converges to some random variable $Z$ with support in $[1,8]

**Proposition 2.** For $\beta R = 1$, the solution of the household problem has $\Pr[\lim_{t \to \infty} c(s^t) = +\infty] = 1$, i.e. consumption diverges to infinity almost surely

**Proof.**
• By the Martingale Convergence Theorem, the marginal utility of consumption converges almost surely to a finite number. (What finite number could depend on the sequence of income realizations).

• Suppose that number is greater than zero. Then consumption converges to a finite number.

• Since after any sequence \( s^t \) the realization of \( y_t \) could be \( y(s_1) \) forever, then the maximum constant consumption that is sustainable under all conceivable future income realizations after history \( s^t \) is \( c(s^{t+n}) = \frac{y(s_1)}{R} + \frac{R-1}{R} x(s^t) \) for all \( n \geq 0 \), i.e. the annuity value of current assets and a stream of future incomes all equal to the lowest possible realization.

• But the household can improve upon this constant consumption by ratcheting up consumption by \( \frac{R-1}{R} (y(s_t) - y(s_1)) \) forever whenever an income level other than \( y(s_1) \) occurs, which contradicts optimality.

• Hence, a time-invariant consumption level does not maximize utility, and future consumption must vary with income shocks. As a result, consumption cannot converge to a finite limit.

• Therefore \( u'(c_t) \) must converge to zero

• Therefore (assuming \( u'(c) > 0 \forall c \), \( c_t \) must diverge to \( \infty \)

(See Chamberlain and Wilson [2000] for a rigorous version of this proof and exact conditions under which it holds)

• Strong precautionary motive: assets also diverge to \( \infty \)

• Does not depend on \( u''(c) \) (but in a way it does)

4.2 The iid-CARA case in detail (without making assumptions on \( \beta \) and \( R \))

• Suppose \( u(c) = -\exp(-\gamma c) \)

• Do not impose a borrowing limit (just a no-Ponzi condition)

• Allow \( c < 0 \)

• Suppose \( y \) is \( iid \)
• Useful properties of CARA:

\[ u'(c) = \gamma \exp(-\gamma c) = -\gamma u(c) \]
\[ u(a+b) = -u(a)u(b) \]
\[ u^{-1}(v) = -\frac{1}{\gamma} \log(-v) \]
\[ \frac{1}{u(c)} = u(-c) \]

• Household solves

\[ V(x) = \max_c u(c) + \beta \sum_{s'} \Pr(s') V(R[x - c] + y(s')) \]

• Guess:

\[ V(x) = Au\left(\frac{R-1}{R}x\right) \]

• Using guess:

\[ V(x) = \max_c u(c) + \beta A \mathbb{E}\left[u\left(\frac{R-1}{R} [Rx - c + y]\right)\right] \]
\[ = \max_c u(c) + \beta A \mathbb{E}\left[u\left((R-1)(x - c) + \frac{R-1}{R} y\right)\right] \]

Let

\[ \alpha \equiv x - c - \frac{1}{R}x \]
\[ \Rightarrow c = \frac{R-1}{R}x - \alpha \]
\[ (R-1)(x-c) = (R-1)\alpha + \frac{R-1}{R}x \]

so

\[ V(x) = \max_\alpha u\left(\frac{R-1}{R}x - \alpha\right) + \beta A \mathbb{E}\left[u\left((R-1)\alpha + \frac{R-1}{R}x + \frac{R-1}{R} y\right)\right] \]
\[ = -u\left(\frac{R-1}{R}x\right) \max_\alpha \left[u(-\alpha) + \beta A \mathbb{E}\left[u\left((R-1)\alpha + \frac{R-1}{R} y\right)\right] \right] \]

• Therefore

\[ A = -\max_\alpha \left[u(-\alpha) + \beta A \mathbb{E}\left[u\left((R-1)\alpha + \frac{R-1}{R} y\right)\right] \right] \] (6)
which verifies the guess because $A$ is a constant.

- Solve by taking FOCs:

\[-u'(-\alpha) + \beta A (R - 1) \mathbb{E} \left[ u' \left( (R - 1) \alpha + \frac{R - 1}{R} y \right) \right] = 0\]

\[u(-\alpha) = \beta A (R - 1) \mathbb{E} \left[ u \left( (R - 1) \alpha + \frac{R - 1}{R} y \right) \right] = 0 \quad (7)\]

- Recall (6):

\[A = -u(-\alpha) - \beta A \mathbb{E} \left[ u \left( (R - 1) \alpha + \frac{R - 1}{R} y \right) \right]\]

and using (7):

\[A = -u(-\alpha) - \frac{u(-\alpha)}{R - 1}\]

\[= -\frac{R}{R - 1} u(-\alpha)\]
• Replace back in the FOC (7)

\[ u(-\alpha) = -\beta Ru(-\alpha)\mathbb{E} \left[ u \left( (R - 1) \alpha + \frac{R - 1}{R} y \right) \right] \]

\[ 1 = -\beta R \mathbb{E} \left[ u \left( (R - 1) \alpha + \frac{R - 1}{R} y \right) \right] \]

\[ = \beta Ru((R - 1) \alpha) \mathbb{E} \left[ u \left( \frac{R - 1}{R} y \right) \right] \]

\[ u(-(R - 1) \alpha) = \beta R \mathbb{E} \left[ u \left( \frac{R - 1}{R} y \right) \right] \]

\[ -(R - 1) \alpha = u^{-1} \left( \beta R \mathbb{E} \left[ u \left( \frac{R - 1}{R} y \right) \right] \right) \]

\[ \alpha = -u^{-1} \left( \beta R \mathbb{E} \left[ u \left( \frac{R - 1}{R} y \right) \right] \right) \]

\[ = \frac{1}{R - 1} \log \left[ \beta R \mathbb{E} \left[ u \left( \frac{R - 1}{R} y \right) \right] \right] \]

\[ = \frac{1}{R - 1} \log (\beta R) + \log \left[ \mathbb{E} \left[ u \left( \frac{R - 1}{R} y \right) \right] \right] \]

\[ = \frac{1}{R - 1} \log (\beta R) - u^{-1} \left( \mathbb{E} \left[ u \left( \frac{R - 1}{R} y \right) \right] \right) \]

(8)

• The policy function for consumption is therefore

\[ c(x) = \frac{R - 1}{R} x - \alpha \]

\[ = \frac{R - 1}{R} x - \frac{1}{\gamma} \log (\beta R) + \frac{u^{-1} \left( \mathbb{E} \left[ u \left( \frac{R - 1}{R} y \right) \right] \right)}{R - 1} \]

• Special case of \( \beta R = 1 \) and no uncertainty about \( y \):

\[ c(x) = \frac{R - 1}{R} x + \frac{y}{R} \]

This is exactly what we found under the PIH!

• The term:

\[ \frac{1}{\gamma} \log (\beta R) \]

adjusts for \( \beta R \neq 1 \)
• The (constant) term:

\[
\frac{u^{-1}\left(\mathbb{E}\left[u \left(\frac{R-1}{R}y\right)\right]\right)}{R-1}
\]

adjusts for the precautionary savings effect.

• \(u\) concave means

\[
\mathbb{E}\left[u \left(\frac{R-1}{R}y\right)\right] < u \left(\frac{R-1}{R} \mathbb{E}y\right)
\]

\[
\Rightarrow u^{-1}\left(\mathbb{E}\left[u \left(\frac{R-1}{R}y\right)\right]\right) < u^{-1}\left(u \left(\frac{R-1}{R} \mathbb{E}y\right)\right) = \frac{R-1}{R} \mathbb{E}y
\]

• Hence, the policy function for consumption has the same slope as under PIH, but a lower intercept. PIH would hold under risk-neutrality.

• Evolution of wealth:

\[
x_{t+1} = R \left(x_t - c(x_t)\right) + y_{t+1}
\]

\[
= R \left(x_t - \frac{R-1}{R} x_t + \alpha\right) + y_{t+1}
\]

\[
= x_t + y_{t+1} + \alpha R
\]

so wealth is a random walk with drift. The drift is given by (8)

• Implication: the variance of the wealth distribution in the population diverges to infinity! There is no invariant distribution.

• Consumption

\[
c_t = \frac{R-1}{R} x_t - \alpha
\]

is also a random walk with drift (since it is a linear function of a random walk process).

• Drift: consumption will tend to increase iff

\[
c < \frac{R-1}{R} x + \frac{\mathbb{E}y}{R}
\]

\[
-\frac{1}{\gamma} \log (\beta R) + \frac{u^{-1}\left(\mathbb{E}\left[u \left(\frac{R-1}{R}y\right)\right]\right)}{R-1} < \frac{\mathbb{E}y}{R}
\]

• It could drift off to infinity even with \(\beta R < 1\), if there is sufficient risk and risk aversion
• Ways out:
  – Life cycle (OLG): people retire/die, no perfect inheritability. This limits the persistence of wealth and consumption inequality.
  – Infinite horizon but:
    * other preferences
    * borrowing constraints

4.3 $\beta R < 1$ and decreasing absolute risk aversion

• Recall that the coefficient of absolute risk aversion is defined as
  $$\gamma(c) = -\frac{u''(c)}{u'(c)}$$

• Now assume that $\beta R < 1$ and
  $$\lim_{c \to +\infty} \gamma(c) = 0$$

• Idea: large precautionary savings effect for low assets, but goes to zero for high assets. This allows for some upper bound on assets and thus a non-degenerate wealth distribution.

• Focus on the iid case for simplicity (but the result is more general)

**Proposition 3.** Assume $\beta R < 1$ and $\lim_{c \to +\infty} \gamma(c) = 0$. Let $c(x)$ be the solution to program (5) and let $x'_{\text{max}}(x) = R \left[ x - c(x) \right] + y(s_n)$ and $x'_{\text{min}}(x) = R \left[ x - c(x) \right] + y(s_1)$. There exists a value $x^*$ such that $x'_{\text{max}}(x) \leq x$ for all $x \geq x^*$.

**Proof.** Below is part of the proof. There are three things missing from it (left as an exercise!) \qed

1. Showing that the borrowing constraint does not bind for sufficiently high $x$
2. Showing that $c(x)$ is increasing
3. Showing that $\lim_{x \to \infty} c(x) = \infty$

For sufficiently high $x$ the borrowing constraint will not bind, so the Euler equation is

$$u'(c(x)) = \beta R \sum_{s'} \Pr(s') u'\left( c \left( R \left[ x - c(x) \right] + y(s') \right) \right)$$

$$= \beta R \sum_{s'} \Pr(s') u'\left( c \left( R \left[ x - c(x) \right] + y(s') \right) \right) u'(c(x'_{\text{max}}(x))) \quad (9)$$
Note that since $c(x)$ is increasing, the following inequalities hold:

\[
1 \leq \frac{\sum_{s'} \Pr(s') u'(c(R[x - c(x)] + y(s'))}{u'(c(x'_{\max}(x)))} \leq \frac{u'(c(x'_{\min}(x)))}{u'(c(x'_{\max}(x)))} \leq \frac{1}{u'(c(x'_{\max}(x)))} \left[ u'(c(x'_{\max}(x))) - \int_{c(x'_{\min}(x))}^{c(x'_{\max}(x))} u''(c) dc \right]
\]

The fact that $\lim_{c \to +\infty} \gamma(c) = 0$ implies that

\[
\lim_{x \to \infty} \frac{\sum_{s'} \Pr(s') u'(c(R[x - c(x)] + y(s'))}{u'(c(x'_{\max}(x)))} = 1 \tag{10}
\]

Replacing (10) in (9) and taking limits:

\[
\lim_{x \to \infty} \left[ u'(c(x)) - \beta Ru'(c(x'_{\max}(x))) \right] = 0
\]

Since $\beta R < 1$, this implies that for high enough $x$,

\[
u'(c(x)) < u'(c(x'_{max}(x)))
\]

and hence

\[
c(x'_{\max}(x)) < c(x),
\]

so consumption will fall for any realization of $s'$. Given that $c(x)$ is increasing, the result follows.

- Consumption does not grow to infinity
- There is an upper bound on wealth
- There is an invariant distribution of wealth for any given individual
- If we think of the population as a collection of independent individuals, there is a steady state wealth distribution
References


