Online Appendix (March 22, 2017) to "Revenue Ranking of Optimally Biased Contests: the Case of Two Players," by Christian Ewerhart

Detailed proof of Lemma A.1. Let $\mu^* = (\mu_1^*, \mu_2^*)$ and $\mu^{**} = (\mu_1^{**}, \mu_2^{**})$ be equilibria in $\mathcal{C}(V_1, V_2, r)$. Then, since μ_1^* is a best response to μ_2^*,

$$p_1(\mu_1^{**}, \mu_2^*)V_1 - E[x_1 |\, \mu_1^{**}] \leq p_1(\mu_1^*, \mu_2^*)V_1 - E[x_1 |\, \mu_1^*], \tag{12}$$

or equivalently,

$$p_1(\mu_1^{**}, \mu_2^*) - p_1(\mu_1^*, \mu_2^*) \leq \frac{E[x_1 |\, \mu_1^{**}] - E[x_1 |\, \mu_1^*]}{V_1}. \tag{13}$$

But winning probabilities add up to one, so that (13) may be written as

$$p_2(\mu_1^*, \mu_2^*) - p_2(\mu_1^{**}, \mu_2^*) \leq \frac{E[x_1 |\, \mu_1^{**}] - E[x_1 |\, \mu_1^*]}{V_1}. \tag{14}$$

Next, since μ_2^{**} is a best response to μ_1^{**},

$$p_2(\mu_1^{**}, \mu_2^*)V_2 - E[x_2 |\, \mu_2^*] \leq p_2(\mu_1^{**}, \mu_2^{**})V_2 - E[x_2 |\, \mu_2^{**}], \tag{15}$$

or equivalently,

$$p_2(\mu_1^{**}, \mu_2^*) - p_2(\mu_1^{**}, \mu_2^{**}) \leq \frac{E[x_2 |\, \mu_2^*] - E[x_2 |\, \mu_2^{**}]}{V_2}. \tag{16}$$

Adding inequalities (14) and (16) up, one finds

$$p_2(\mu_1^*, \mu_2^*) - p_2(\mu_1^{**}, \mu_2^{**}) \leq \frac{E[x_1 |\, \mu_1^{**}] - E[x_1 |\, \mu_1^*]}{V_1} + \frac{E[x_2 |\, \mu_2^*] - E[x_2 |\, \mu_2^{**}]}{V_2}. \tag{17}$$

Repeating the exercise with the roles of μ^* and μ^{**} exchanged shows that

$$p_2(\mu_1^{**}, \mu_2^{**}) - p_2(\mu_1^*, \mu_2^*) \leq \frac{E[x_1 |\, \mu_1^*] - E[x_1 |\, \mu_1^{**}]}{V_1} + \frac{E[x_2 |\, \mu_2^{**}] - E[x_2 |\, \mu_2^*]}{V_2}, \tag{18}$$

so that (17) is an equality. But then, also all the inequalities on the way, such as (12) and (15), as well as their counterparts with μ^* and μ^{**} exchanged, must also be equalities. Therefore, $\Pi_1(\mu_1^*, \mu_2^{**}) = \Pi_1(\mu_1^{**}, \mu_2^{**}) \geq \Pi_1(\mu_1, \mu_2^{**})$ and $\Pi_1(\mu_1^{**}, \mu_2^*) = \Pi_1(\mu_1^*, \mu_2^*) \geq \Pi_1(\mu_1, \mu_2^*)$ for any $\mu_1 \in \mathcal{M}_1$, and $\Pi_2(\mu_1^*, \mu_2^{**}) = \Pi_1(\mu_1^*, \mu_2^*) \geq \Pi_1(\mu_1^*, \mu_2)$ and $\Pi_2(\mu_1^{**}, \mu_2^*) = \Pi_2(\mu_1^{**}, \mu_2^{**}) \geq \Pi_1(\mu_1^{**}, \mu_2)$ for any $\mu_2 \in \mathcal{M}_2$, so that both (μ_1^*, μ_2^{**}) and (μ_1^{**}, μ_2^*) are equilibria as well. $\square$