A short and intuitive proof of Marshall's Rule*

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Summary. When the price of an input factor to a production process increases, then the optimal output level declines and the input is substituted by other factors. Marshall's rule is a formula that determines the own-price elasticity for one factor as a weighted sum of the elasticities of output market demand and factor substitution. This note offers a proof for Marshall's rule that is significantly shorter and somehow more intuitive than existing derivations.

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1. Introduction

Under the neoclassical paradigm of a profit maximizing firm, higher prices for an input factor to a production process will typically imply effects on the optimal level of output, as well as substitution effects by other factors. For example, when Indonesia's computer chip industry suffered from a series of earth quakes in 1999, this led temporarily to smaller main storages in personal computers, and also to a stronger substituton of these chips by those manufactured in other countries.

This intuitive relationship can be captured in terms of a decomposition of the own-price elasticity of a factor's derived demand. In his *Principles*, Alfred Marshall [3] formulated four rules on the determinants of this price elasticity and gave a mathematical derivation for a special case. Hicks [1] was first in providing a full-fledged mathematical treatment. He presented an equation that yields the own price elasticity of derived demand in terms of elasticities of substitution and original demand.

While this equation, which is referred to as "Marshall's rule," is economically very intuitive, existing derivations are surprisingly involved even in the simplest case of two factors and perfectly elastic supply of the second input factor (see, e.g., [2], Chapter 9 and Appendix 8, or [4]). Moreover, typically proofs are based on second-order derivatives which do not become part in the rule's final form nor in its economic interpretation.

The key innovation of the proof given in this note is it to view the required level for one factor as an implicitly defined function of output level and $n-1$ input factor proportions. The own-price elasticity of demand for this factor then decomposes, as a natural consequence of the chain rule, into n components, which can be shown to correspond to one output effect and $n-1$ substitution effects.

The approach thereby circumvents the use of second-order derivatives, is shorter, and probably also more intuitive than existing proofs.[1]

2. Formal discussion

Consider a market given by the demand function $y = y(p)$, with elasticity of demand

$$\varepsilon_d = \frac{\partial \log y}{\partial \log p}. \tag{1}$$

Firms operating in the market possess a (non-trivial) constant returns to scales (CRS) production function

$$y = f(x_1, x_2), \tag{2}$$

with input factors x_1, x_2 (the given proof extends straightforwardly to the case of $n \geq 2$ factors). Let $h_i(w_1, w_2, y)$ be the Hicksian demand for factor i, where w_1 and w_2 denote the prices for the input factors. As the technology is CRS, it is true that $h_1(w_1, w_2, y)/h_2(w_1, w_2, y)$ is a function of w_1/w_2 only. This function is denoted by $q = q(w_1/w_2)$. Then the direct elasticity of substitution (cf. [6]) is defined as

$$\sigma = -\frac{\partial \log q}{\partial \log(w_1/w_2)}. \tag{3}$$

For any pair of prices (w_1, w_2), an equilibrium under perfect competition is a triple (x_1, x_2, p), where profit maximizing firms choose factor quantities $x_1 = x_1(w_1, w_2)$ and $x_2 = x_2(w_1, w_2)$, the output market clears, and profits are zero. Then the own price elasticity of input x_1 is defined as

$$\varepsilon_{11} = \frac{\partial \log x_1}{\partial \log w_1}. \tag{4}$$

[1] In more technical terms, our approach is a change of variables from affine to projective coordinates.

Marshall's Rule: *The own-price elasticity of derived demand for factor 1 decomposes in an output and a substitution effect. More precisely,*

$$\varepsilon_{11} = s_1 \varepsilon_d - (1 - s_1)\sigma, \tag{5}$$

where $s_1 = w_1 x_1/(w_1 x_1 + w_2 x_2)$ *is the share of total production costs accruing to factor 1.*

Proof. Let $x_1 = g(y, q)$ denote the input quantity x_1 necessary to produce output level y at a given input factor proportion $q = x_1/x_2$ (note that g exists and is differentiable by the implicit function theorem). Then, by the chain rule,

$$\varepsilon_{11} = \frac{\partial \log g}{\partial \log y}\frac{\partial \log y}{\partial \log w_1} + \frac{\partial \log g}{\partial \log q}\frac{\partial \log q}{\partial \log w_1}. \tag{6}$$

Since the technology is CRS, we obtain $\partial \log g/\partial \log y = 1$ and $\partial \log q/\partial \log y = 0$. Hence, as w_2 is exogenous, with the equilibrium price $p = p(w_1, w_2)$,

$$\varepsilon_{11} = \frac{\partial \log p}{\partial \log w_1}\varepsilon_d - \frac{\partial \log g}{\partial \log q}\sigma. \tag{7}$$

It remains to calculate the two derivatives in (7). For the first, let $c = c(w_1, w_2, y)$ denote the cost function. Now, the zero profits assumption implies $py = c$, hence, by Shepard's Lemma (cf., e.g., [7], p. 74), and profit maximization,

$$\begin{aligned}\frac{\partial \log p}{\partial \log w_1} &= \frac{\partial \log(py)}{\partial \log w_1} - \frac{\partial \log y}{\partial \log w_1} \\ &= \frac{w_1}{c}\left\{\frac{\partial c}{\partial w_1} + \frac{\partial c}{\partial y}\frac{\partial y}{\partial w_1}\right\} - \frac{w_1}{y}\frac{\partial y}{\partial w_1} \\ &= \frac{w_1}{c}\left\{x_1 + p\frac{\partial y}{\partial w_1}\right\} - \frac{w_1}{y}\frac{\partial y}{\partial w_1} \\ &= s_1.\end{aligned} \tag{8}$$

The second derivative in (7) is determined as follows. Differentiating the definitory equation for g, i.e., $f(x_1, x_1/q) = y$ with respect to q yields

$$f_1\frac{\partial g}{\partial q} + \frac{f_2}{q}\left\{\frac{\partial g}{\partial q} - \frac{x_1}{q}\right\} = 0, \tag{9}$$

where f_i denotes the partial derivative of f with respect to x_i. Using the first-order condition $f_1/f_2 = w_1/w_2$, this implies

$$\frac{\partial \log g}{\partial \log q} = \frac{1}{x_2}\frac{\partial g}{\partial q} = \frac{f_2/q}{f_1 + f_2/q} = 1 - s_1, \tag{10}$$

and thereby proves the assertion. Q.E.D.

Up to this point, we considered a special case of Marshall's rule characterized by perfectly elastic supply of factors $2, ..., n$. The more general case (cf. Sato and Koizumi [5]) follows along similar lines. We briefly indicate the derivation for $n = 2$. Start with (6). The output effect becomes

$$\frac{\partial \log p}{\partial \log w_1} = s_1 + (1 - s_1)\frac{\partial \log w_2}{\partial \log w_1}. \tag{11}$$

Because of CRS,

$$\frac{\partial \log q}{\partial \log w_1} = \sigma(1 - \frac{\partial \log w_2}{\partial \log w_1}). \tag{12}$$

Differentiating the zero-profit assumption

$$w_1 x_1 + w_2 x_2 = py \tag{13}$$

with respect to w_1, using (11), and rearranging yields

$$\frac{\partial \log w_2}{\partial \log w_1} = \frac{s_1}{1 - s_1}\frac{\varepsilon_d - \varepsilon_{11}}{\varepsilon_{22} - \varepsilon_d}, \tag{14}$$

where

$$\varepsilon_{22} = \frac{\partial \log x_2}{\partial \log w_2} \tag{15}$$

denotes the supply elasticity for factor 2. Using (14) in (11) and (12), this completes the derivation of the more general form of Marshall's rule (cf. Hicks, 1963):

$$\varepsilon_{11} = \frac{\sigma(\varepsilon_d - \varepsilon_{22}) - s_1\varepsilon_{22}(\varepsilon_d - \sigma)}{\varepsilon_d - \varepsilon_{22} - s_1(\varepsilon_d - \sigma)} \tag{16}$$

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