Iterated Weak Dominance in Strictly Competitive Games of Perfect Information[1]

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ABSTRACT

We prove that any strictly competitive perfect-information two-person game with *n* outcomes is solvable in *n-1* steps of elimination of weakly dominated strategies – regardless of the length of the game tree. The given bound is shown to be tight using a variant of Rosenthal's centipede game. *Journal of Economic Literature* Classification Numbers: C70, C72.

1. INTRODUCTION

In this paper, it is shown that any zero-sum game of perfect information with *n* outcomes is solvable by *n-1* iterations of eliminating weakly dominated strategies – regardless of the length of the game.

The intuition is as follows. Suppose that player *i*'s maximum possible payoff is u_i^{max}. Then, if *i* has a strategy ensuring u_i^{max}, this strategy is dominant and the game solves in one step. On the other hand, if *i* has no such "winning" strategy, then on any path that leads to u_i^{max}, player *j* must take a "surrendering" move at some decision node that allows *i* to ensure u_i^{max} in the ensuing subgame. As ensuring u_i^{max} is the sole undominated strategy for *i* in the subgame, it turns out that any strategy by *j* that may entail a surrendering move is eliminated by two steps of eliminating weakly dominated strategies. Hence, after two steps of elimination, it is no longer feasible to reach any terminal node with payoff u_i^{max}. Since the same is true for player *j*'s maximum possible payoff u_j^{max}, the first two steps of elimination must delete the two extreme outcomes in the game (unless either *i* or *j* had a winning strategy).

This argument can be iterated in a straightforward way. Specifically, when the number of outcomes *n* is odd, i.e., equal to *2m+1* for some *m*, then one needs at most *2m=n-1* steps to arrive at a trivial game. When *n* is even, i.e., equal to *2m* for some *m*, then the game has at most two outcomes left after *2(m-1)* steps of elimination, so the game solves in at most *2(m-1)+1=n-1* steps. Thus, in both cases, *n-1* steps are sufficient.

The application of iterated weak dominance to games of perfect information has been studied before. In [5], Gale suggests an equilibrium selection procedure in finite perfect-information games using iterated dominance arguments. A key contribution is [9], in which

Moulin formulates a condition on perfect-information games implying dominance-solvability, i.e., that the iterated elimination of all weakly dominated strategies yields a game in which all strategy combinations induce the same outcome. The proof uses the fact that backward induction reasoning eliminates weakly dominated strategies in every step, though not necessarily all of them. The condition formulated by Moulin guarantees that backward induction yields a singleton outcome, but also that the order of elimination does not matter, implying that iterated weak dominance leads to the same unique outcome as backward induction. In [6], Gretlein discovers a gap in Moulin's proof and closes it using results of a companion paper [7]. A frequently cited unpublished working paper [13] by Rochet seems to contain results similar to those in [7] (see [4] and [11]). The careful reader of this literature will notice that in [10], Moulin attributes the above-mentioned result in [9] to Harold Kuhn. The present paper builds upon and extends the main result in [3], which says that any zero-sum perfect-information game with three outcomes is solvable in two steps.[2]

The rest of this paper is structured as follows. Section 2 recalls some standard terminology. In Section 3, we present the concept of a surrender strategy and apply it to derive the main result. The tightness of the upper bound is shown in Section 4. The appendix contains technical proofs.

2. NOTATION

Consider a finite perfect-information game G (for definitions of the standard concepts used, we refer the reader to Binmore [2]). Players are denoted by $i, j \in \{1,2\}$, where $j \neq i$. Let X_i

[2] After this work was completed, we learned of a recent paper by Shimoji [14] that addresses a similar question and obtains results similar to ours.

denote the set of i's nodes in G, let Z be the set of terminal nodes, and let $X=X_i \cup X_j \cup Z$. A *move* (x,y) is a pair of nodes such that y is an immediate successor of x. A *strategy* s for player i specifies moves at any node $x \in X_i$. Denote a typical strategy for j by t, and the set of strategies for players i and j by S and T, respectively. We say that $s \in S$ *reaches* x if there is a $t \in T$ such that the path generated by the profile (s,t) contains x. The *utility* of player i is $u_i(s,t)=u_i(z)$, where z is the terminal node contained in the path generated by (s,t).

Throughout the paper, we assume that G is *zero-sum*, i.e. that $u_i(s,t)=-u_j(s,t)$ for all s and t. In the present context, this is tantamount to saying that the game is *strictly competitive*, i.e. that the elements of the set of outcomes $\{(u_i(s,t),u_j(s,t)) | s \in S,\ t \in T\}$ can be labeled as $a_1,...,a_n$ so that player i prefers an outcome with lower index over an outcome with higher index, and vice versa for player j. For any normal-form game N with utility $u_i(s,t)$, denote by $v_i(N)= \max_s \min_t u_i(s,t)$ player i's *value* or *security level* in N. Write $N=N(G)$ for the normal form of G, and $v_i(G):= v_i(N(G))$. It is well known ([8,15]) that $v_i(G)=-v_j(G)$. We will refer to this result as the *minmax theorem for PI games*. For a node $x \in X$, denote by $G(x)$ the subgame starting at x, and by $v_i(x)$ player i's value in $G(x)$. A strategy s is a *maximin strategy* in G if $u_i(s,t) \geq v_i(G)$ for all $t \in T$. We will write u_i^{max} for i's maximum possible payoff $\max\{u_i(s,t) | s \in S, t \in T\}$ in N, and similarly u_i^{min} for $\min\{u_i(s,t) | s \in S, t \in T\}$. A strategy s is *winning* if $u_i(s,t)=u_i^{max}$ for all $t \in T$.

Let $T_0 \subseteq T$ be a set of strategies for player j. A strategy $s \in S$ is *(weakly) dominated* by $s' \in S$ *with respect to* T_0 if $u_i(s,t) \leq u_i(s',t)$ for all $t \in T_0$, and $u_i(s,t') < u_i(s',t')$ for some $t' \in T_0$. A strategy $s \in S$ is (weakly) dominated *in* N if it is dominated by some $s' \in S$ with respect to T. Denote by $D(N)$ the normal-form game resulting from N by eliminating all strategies, for i and j, that are dominated in N. As usual, write $D^0(N):=N$ and $D^k(N):=D(D^{k-1}(N))$ for the game resulting from k times repeated elimination of weakly dominated strategies. Abusing notation, we write $D(S)$ and $D^k(S)$ for the set of i's strategies in $D(N)$ and $D^k(N)$, respectively (analogously for player j). We say that a normal-form game N has n outcomes if the image of the mapping (u_1,u_2) in $\hat{A}^2$ has cardinality n. The game N is *trivial* if it possesses only one outcome.

3. SURRENDER STRATEGIES

The main result of the paper says that any zero-sum perfect-information game G with n outcomes is dominance solvable in $n-1$ steps. For an example, consider Figure 1. Here, a one-time elimination of weakly dominated strategies reduces G to G'. While this simplifies the game, the reader will also notice that the number of outcomes remains unchanged at three. It is therefore not possible to employ a direct induction argument on the number of outcomes. However, note that an additional step of elimination applied to G' renders a game with just one outcome.

It turns out that this is true in general. More precisely, it will be shown below that, in all relevant cases, two steps of elimination reduce the number of outcomes by at least two. In fact, two steps of eliminating dominated strategies delete "extreme" outcomes, characterized by the property that they give either player 1 or player 2 her most preferred payoff. This is why the following definition is of interest. Assume that player j does not possess a winning strategy in N. Then a *surrender* strategy for player i in N is a strategy s with the property that there exists a strategy t for player j such that $u_j(s,t)$ is the maximum possible utility for j in N.

- Figure 1 here -

Consider again the example depicted in Figure 1. Playing left at the initial move means a surrender for player 1 since, following 1's move, player 2 may end the game in her optimal outcome.

Lemma 1. *Let G be a strictly competitive game of perfect information with normal form N. Assume that player j does not possess a winning strategy in* $D^k(N)$*, for some* $k \geq 0$*. Then any surrender strategy in* $D^k(N)$ *for player i is eliminated by two steps of iterated dominance, i.e., it is not contained in* $D^{k+2}(N)$*.*

The mechanics for this result were explained in the introduction (the proof can be found in the appendix). Roughly speaking, the first step of elimination deletes all non-winning strategies for j in subgames where j possesses a winning strategy. Then, the second step deletes all strategies for i which enable j to reach one of these subgames, i.e., the second step deletes all surrender strategies.

Lemma 2. *Let G be a zero-sum game of perfect information, and* $k \geq 0$*. Assume that* $D^k(N)$ *possesses only two outcomes. Then* $D^{k+1}(N)$ *is trivial.*

Proof. By Lemma A.2 in the Appendix, the minmax theorem for PI games is valid also for the game $D^k(N)$ (which is not necessarily of perfect information). That is, we have $v_i(D^k(N))+v_j(D^k(N))=0$. As $v_i(D^k(N))$ can attain only one of two possible values, either i or j must possess a winning strategy in $D^k(N)$. Assume without loss of generality that i has a winning strategy. Then, as any strategy for i that is not winning in $D^k(N)$ is weakly dominated, the game $D^{k+1}(N)$ has only one outcome.¶

Given these preparations, the upper bound can be derived as follows. When no player has a winning strategy, the elimination of surrender strategies reduces the number of outcomes by two. Hence, a fortiori, the two-fold elimination of dominated strategies also reduces the number of outcomes by two. Iterating this argument yields our main result.

Theorem 1. *Let G be a strictly competitive perfect-information game with n outcomes. Then n-1 iterations of eliminating weakly dominated strategies reduce G to a trivial game.*

Proof. The assertion follows by induction from Lemma 1 if n is odd. If n is even, apply Lemma 1 $(n-1)/2$ times, and apply Lemma 2 afterwards. ¶

The proof of this result (especially that of Lemma 1) is more complicated than in the case $n=3$ mainly because of two reasons. Firstly, two steps of elimination will typically delete not only surrender strategies, but other strategies as well. This makes it difficult to use a more direct induction argument. The second complication is that the elimination of weakly dominated strategies in a game of perfect information does not necessarily render a perfect information game (see [1] for an example), which also precludes the use of a simpler induction argument.

4. TIGHTNESS OF THE UPPER BOUND

For a general finite strictly competitive perfect-information game with n outcomes, it does not suffice to perform less than $n-1$ steps of elimination. An example is G_n in Figure 2, which is a variant of Rosenthal's centipede game (see [12]). The outcomes are $a_1,...,a_n$, where player 1 prefers an outcome with lower index over an outcome with higher index, and vice versa for player 2. Consider first the case that n is even (left side of Figure 2). We claim that for player 1, only the strategy s_0 that goes straight through to the outcome a_n is weakly dominated, and that for player 2, no strategy is dominated. To see why, note that in order to determine the payoff of a strategy pair, it suffices to distinguish between strategies in terms of the first "exit point". Then, if a strategy s' exhibits a performance against some strategy t that is strictly different from that of another strategy s (which is a necessary condition for a weak dominance relationship), then this means that the exit point of t is either between or behind those of s and s'. However, if this is the case, then the superiority relation between s and s' is reversed when the exit point of t is moved, either from "between" to "behind," or vice versa. This is always possible unless either s or s' does not exit at all, i.e., is equal to s_0. An analogous argument applies when n is odd. It follows that, by identifying certain payoff-equivalent strategies, we can "reduce" $D(N(G_n))$ to $N(G_{n-1})$. As this is true for all n, and the operator $D(.)$ preserves the above reducibility relation, the game $D^{n-2}(N(G_n))$ may be reduced to $N(G_2)$, and therefore is not trivial. We have proved:

Theorem 2. *For any $n \geq 2$, there exists a zero-sum perfect-information game G_n that is dominance solvable in n-1 steps, but not in n-2 steps.*

- Figure 2 here -

APPENDIX: TECHNICAL PROOFS

Lemma A.1. *Let* $s \in S$. *Then there is an* $s' \in D^k(S)$ *such that* $u_i(s',t) \geq u_i(s,t)$ *for all* $t \in D^k(T)$.

Proof. By induction on k. If $k=0$, let $s':=s$. Assume the assertion is true for some $k \geq 0$. Let $s \in S$. By the induction hypothesis, there is an $s'' \in D^k(S)$ such that $u_i(s'',t) \geq u_i(s,t)$ for all $t \in D^k(T)$. Any element of $D^k(S)$ is either undominated or dominated by an undominated strategy (since the relation of weak dominance is acyclic). Therefore we may choose $s' \in D^{k+1}(S)$ such that $u_i(s',t) \geq u_i(s'',t)$ for all $t \in D^k(T)$. Putting these inequalities together gives $u_i(s',t) \geq u_i(s,t)$ for all $t \in D^k(T)$, which a fortiori holds for all $t \in D^{k+1}(T)$.¶

Lemma A.2. *If N is the normal form of a zero-sum perfect-information game, we have* $v_i(D^k(N))=v_i(N)$ *for all* $k \geq 0$.

Proof. By Lemma A.1,

$$v_i(D^k(N)) \geq v_i(N). \qquad \text{(i)}$$

As N is the normal form of a perfect information game, the minmax theorem implies

$$v_i(N)+v_j(N)=0. \qquad \text{(ii)}$$

Thus, from (i),

$$v_i(D^k(N))+v_j(D^k(N)) \geq 0. \quad \text{(iii)}$$

On the other hand, clearly $v_i(D^k(N))+v_j(D^k(N)) \leq 0$ (otherwise, the players could collectively achieve more than a zero payoff in $D^k(N)$). So (iii) is satisfied with equality, and therefore so must (i) be.¶

Proof of Lemma 1. Let $s \in D^k(S)$ be a surrender strategy in $D^k(N)$. Then, by definition, there exists a $t \in D^k(T)$ so that $u_j(s,t)=u_j^{max}$, where u_j^{max} denotes the maximum possible payoff for j in $D^k(N)$. In fact, without loss of generality we may choose $t \in D^{k+1}(T)$. By assumption, $v_j(D^k(N))<u_j^{max}$. Therefore, by Lemma A.2, $v_j(G)<u_j^{max}$. Hence, there is a move (x,y) on the path generated by (s,t) so that

$$v_j(x)<u_j^{max} \leq v_j(y). \qquad (*)$$

As j cannot increase her value by making a move, player i is called upon to play at x. Consider strategy $s' \in S$ for i, equal to s outside $G(x)$, and equal to some maximin strategy in $G(x)$. Assume $s \in D^{k+1}(S)$ (otherwise there is nothing to show). We claim that s' dominates s with respect to $D^{k+1}(T)$. By Lemma A.1, this is enough to prove the assertion. First we show that $u_i(s',t') \geq u_i(s,t')$ for all $t' \in D^{k+1}(T)$. Assume to the contrary that

$$u_i(s',t')<u_i(s,t') \qquad (**)$$

for some $t' \in D^{k+1}(T)$. Note that, because of (**), both (s',t') and (s,t') must reach x. Hence, as s' is maximin for i in $G(x)$, we have $u_i(s',t') \geq v_i(x)$. Then, again because of (**), we have

$$u_i(s,t')>v_i(x). \qquad (***)$$

Now, (s,t') reaches y, as i moves at x. But then, we claim, t' is dominated in $D^k(N)$. For this, define $t'' \in T$ to be equal to t' outside of $G(y)$ and equal to some maximin strategy in $G(y)$. Then $u_j(s'',t'') \geq u_j(s'',t')$ for all $s'' \in D^k(S)$, because if (s'',t'') does not reach y, then $u_j(s'',t'')=u_j(s'',t')$, and if (s'',t'') reaches y, then $u_j(s'',t'') \geq v_j(y) \geq u_j^{max} \geq u_j(s'',t')$ from (*)). As (s,t'') reaches y, the minmax theorem for PI games, (*) and (***) imply

$$u_j(s,t'') \geq v_j(y)>v_j(x)=-v_i(x)>-u_i(s,t')=u_j(s,t').$$

Thus, t' is dominated by $t'' \in T$ with respect to $D^k(S)$. But then, by Lemma A.1, strategy t' is dominated in $D^k(N)$, which is a contradiction to our assumption that $t' \in D^{k+1}(T)$. Hence (**)

cannot hold. It remains to be shown that s' is sometimes strictly better than s with respect to $D^{k+1}(T)$. We claim that $u_i(s',t)>u_i(s,t)$. This is because (s',t) reaches x, and therefore, by (*) and the minmax theorem for PI games,

$$u_i(s',t) \geq v_i(x)=-v_j(x)>-u_j^{max}=-u_j(s,t)=u_i(s,t).$$

Hence, summing up, $s' \in S$ dominates s with respect to $D^{k+1}(T)$. This proves the assertion.¶

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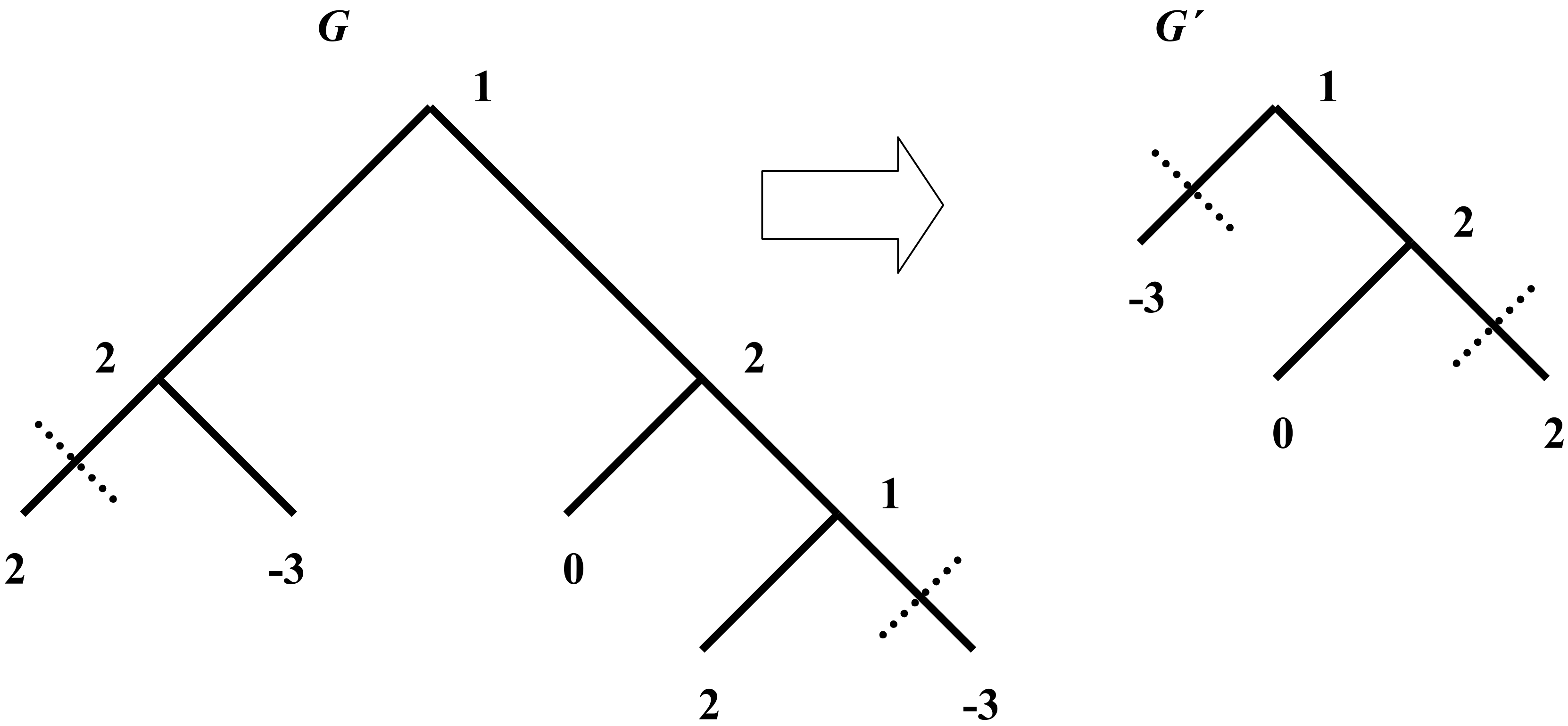

Figure 1. The elimination of dominated strategies need not reduce the number of outcomes (payoffs are for player 1; the arrow and the dotted lines represents the one-time elimination of dominated strategies)

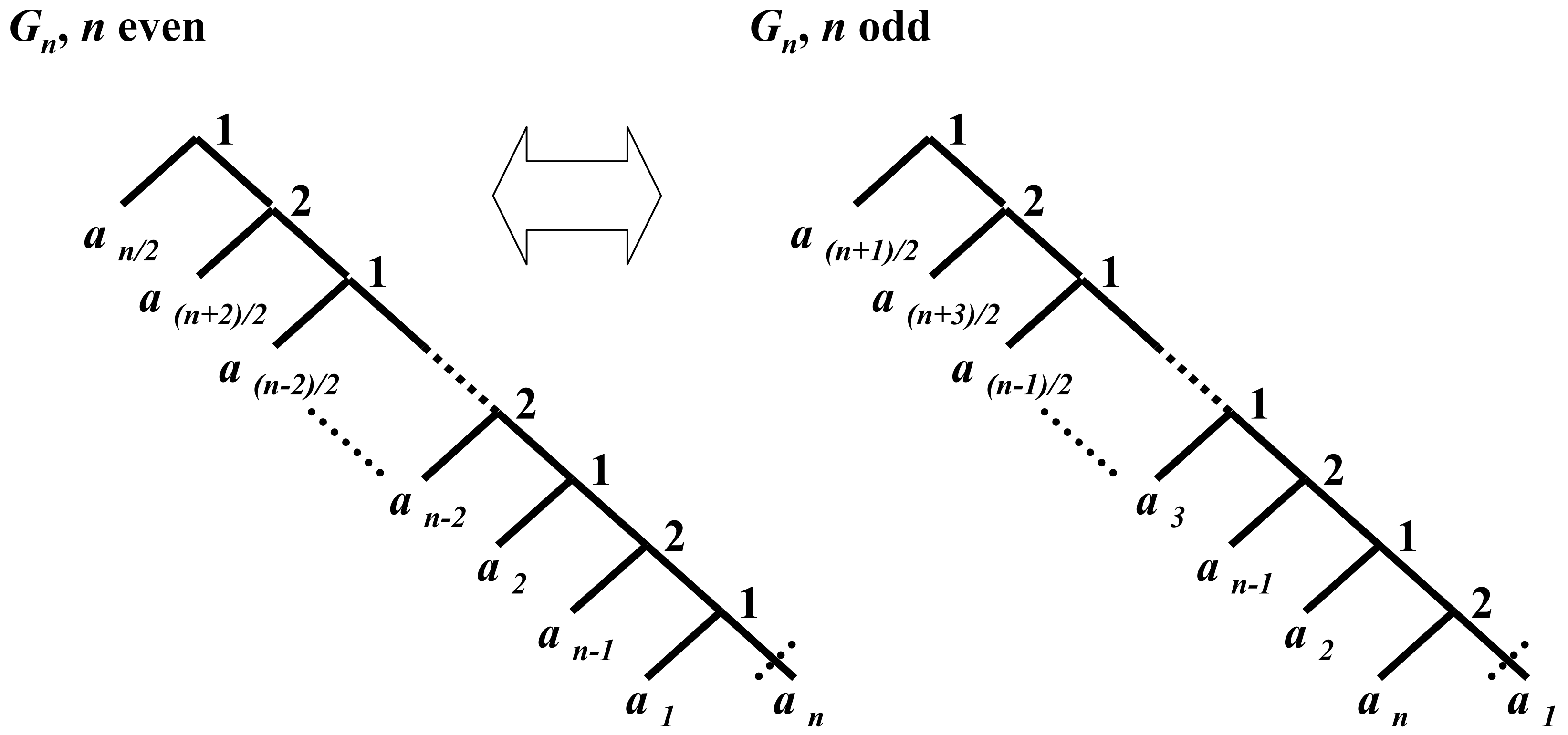

Figure 2. A strictly competitive game of perfect information with n ³ *2* outcomes that is dominance solvable in precisely *n-1* steps; the one-time elimination of dominated strategies reduces G_n to G_{n-1} (implying a change of sides in the diagram)