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Abstract

Under the assumption of normally distributed returns, we analyze whether the Cumulative Prospect Theory of Tversky and Kahneman (1992) is consistent with the Capital Asset Pricing Model. We find that in every financial market equilibrium the Security Market Line Theorem holds. However, under the specific functional form suggested by Tversky and Kahneman (1992) financial market equilibria do not exist. We suggest an alternative functional form that is consistent with both, the experimental results of Tversky and Kahneman and also with the existence of equilibria.

Keywords: Capital Asset Pricing Model, Prospect Theory.

JEL Classification Numbers: C 62, D 51, D 52, G 11, G 12.

SSRN Classification: Behavioral Finance; Capital Markets: Asset Pricing and Valuation.

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1 Introduction

Jagannathan and Wang (1996) praise the Capital Asset Pricing Model (CAPM) with the words: “The CAPM is widely viewed as one of the two or three major contributions of academic research to financial managers during the post-war era.” The commonly used derivations of the CAPM from first principles like utility maximization and return distributions are, however much less accepted in our profession. Conventional wisdom, as it shows up in our textbooks (see for example Copeland and Weston (1998)), usually derives the CAPM from the expected utility hypothesis and from normally distributed returns. Ever since it was axiomatically founded by von Neumann and Morgenstern (1944) and Savage (1954), the expected utility assumption has been under severe fire as a descriptive theory of investors‘ behavior. Allais (1953), Ellsberg (1961) and Kahneman and Tversky (1979) are three prominent cornerstones of this critique. As De Bondt (1999) has recently put it: “For at least 40 years, psychologists have amassed evidence that ’economic man (Edwards, The Theory of Decision-Making, 1954) - is very unlike a real man’ and that reason - for now, defined by the principles that underlie expected utility theory, Bayesian learning, and rational expectations - is not an adequate basis for a descriptive theory of decision making.”

In 2002 the work of Kahneman and Tversky has been rewarded the Nobel price in economics for providing an alternative to expected utility: prospect theory – a theory that is consistent with the psychology of the investor. Kahneman and Tversky (1979) is the seminal paper on prospect theory. In Tversky and Kahneman (1992) they have suggested to change their theory in one important aspect. Instead of using distortions of probabilities they preferred to use distortions of the cumulative distribution function because this will keep consistency of investors‘ decisions with first order stochastic dominance. In this paper we will focus on the cumulative prospect theory (CPT). Kahneman and Tversky‘s prospect theory deviates from the expected utility hypothesis in four important aspects.

- Investors evaluate assets according to gains and losses and not according to final wealth.
- Investors dislike losses by a factor of 2.25 as compared to their liking of gains.
- Investors‘ von Neumann-Morgenstern utility functions are s-shaped with turning point at the origin.
- Investors probability assessments are biased in the way that extremely small probabilities (extremely high probabilities) are over- (under-) valued.

We demonstrate that although prospect theory deviates from the expected utility hypothesis in these important directions still prospect theory is consistent with Mean-Variance Analysis and the Security Market Line (SML) Theorem, provided one keeps the assumption of normally distributed returns. Hence, prospect theory could be seen as a behavioral foundation of these two important features of the CAPM. However, under the specific functional forms suggested by Tversky and Kahneman (1992) financial market equilibria do not exist. Therefore, we suggest alternative functional forms consistent with the results of Tversky and Kahneman, for which equilibria do exist.

Our paper may help to explain, why behavioral finance has discovered difficulties to pin down behavioral factors that should replace or complement the market portfolio. Moreover, our paper may help to explain the "The Paradox of Asset Prices", as Bossaert (2002) has dubbed it, according to which individual behavior in laboratory experiments contradicts the expected utility hypothesis while the laboratories market prices satisfy the CAPM.

The rest of this paper is organized as follows. In the next section we will outline the standard CAPM-model with exogenously given riskfree rate of return as presented by Sharpe (1964). The mathematical approach, as it is now standard for the CAPM, is taken from Duffie (1988). In Section 3 we demonstrate that the CPT of Tversky and Kahnemann (1992) leads to the mean-variance principle, Tobin Separation and the Mutual Fund Theorem. Hence the security market line theorem of the CAPM holds. In Section 4 we then show that with the specific functional forms for the utility function suggested by Tversky and Kahnemann (1992) no equilibria exist. Finally, we propose an alternative functional form for the utility index for which equilibria do exist.

2 The Model

The description of the model follows Duffie (1988, section I.11). Let $(M, \mathcal{M}, \eta)$ be a probability space. Consider $\mathcal{L}$, the space of real–valued measurable functions on $(M, \mathcal{M}, \eta)$. We endow $\mathcal{L}$ with the scalar product $x \cdot y := \int_M x(m)y(m)d\eta$ and with the norm $\|x\| = \sqrt{(x \cdot x)}$. The consumption set will be the subset of $\mathcal{L}$ with finite norm, $L^2(\eta) = \{x \in \mathcal{L} \mid \|x\|^2 < \infty\}$[1]. The price space is also $L^2(\eta)$. Denote the expectation of a portfolio x by $\mu(x) := \int_M x(m)\eta(dm)$ and the covariance of $x, y \in L^2(\eta)$ by $cov(x, y) = \mu(xy) - \mu(x)\mu(y)$. The standard deviation of x is accordingly $\sigma(x) := \sqrt{cov(x, x)}$.

Let the marketed subspace, X, be generated as the span of $(A_j)_{j=0,1,...,J}$, a

[1] $L^2(\eta)$ is the set of equivalent classes with respect to the equivalence relation $x \sim y \Leftrightarrow \eta(x \neq y) = 0$. $L^2(\eta) = L^2(\pi)$ for all $\pi \sim \eta$.

collection of securities in $L^2(\eta)$, one of which is the riskless asset $1\!\!1$. To nail down the notation, say asset 0 is the riskless asset, $A_0 = 1\!\!1$. With respect to the riskless asset every payoff x in X can be decomposed $x = x_\perp + x_\parallel$ into one part $x_\parallel$ collinear to $1\!\!1$ and one part $x_\perp$ orthogonal to $1\!\!1$. Of course, orthogonality is meant with respect to the scalar product $\cdot$ just defined.

Assumption 1 (Asset Payoffs)

Asset payoffs $A_j \in L^2$ are normally distributed, i.e. $A_j \sim N(\mu_j, \sigma_j)$, $j = 1, .., J$. The supply of risky asset $j = 1, \ldots, J$ is exogenously given and denoted by $\bar{\theta}_j > 0$. The riskfree asset is in elastic supply, with exogenously given price $\frac{1}{1+r}$, where r is the riskfree rate of return. The market portfolio is the sum of all available risky assets, i.e. $\omega = \sum_{j=1}^{J} A_j \, \bar{\theta}_j$. It is assumed that the market portfolio has positive expectation and variance, i.e. $\mu(\omega) > 0$ and $\sigma^2(\omega) > 0$. We say that X has a Hamel basis of jointly normal random variables.

There are $i = 1, ...I$ investors, also called agents or consumers. They are initially endowed with wealth $w^i > 0$. The numbers θ^i_j denote the amount of security j held by agent i, q_j denotes the j-th security price. Thus, when trading these securities, the agent can attain the consumption plan $x = \sum_{j=0}^{J} A_j \, \theta^i_j$ where θ^i satisfies the budget restriction (i.e. $\sum_{j=0}^{J} q_j \, \theta^i_j \leq w^i$).

Agents evaluate consumption plans according to prospect theory utility functions. The first principle of prospect theory is that agents do not evaluate utility according to some utility function $\hat{U}^i(x)$ on final wealth, but they evaluate portfolio choices, using some utility function U^i, based on gains and losses, i.e. based on *changes* in wealth. This can well be accommodated by using the transformations $\hat{U}^i(x) = U^i(x - \beta w^i 1\!\!1)$ and $U^i(\Delta x) := \hat{U}^i(\Delta x + \beta w^i 1\!\!1)$. Note that we did introduce a time preference $\beta > 1$ into the utility function. Hence agents do discount future payoffs. That is to say an investment opportunity has produced a gain only if it has generated sufficient payoffs to compensate for the delay in delivering payoffs. We assume that $\beta = 1 + r$, i.e. investors evaluate gains and losses with respect to the riskfree investment. Choosing the riskfree rate of return as reference point means to frame decisions with respect to excess returns, which is in the spirit of the security market line theorem. Given the initial wealth and the time preference, every assumption on the utility function U^i translates to an according assumption on $\hat{U}^i$ and vice versa. Since one of the assets is the riskless bond, $1\!\!1 \in$ X, the changes of wealth $\Delta x = x - \beta w^i 1\!\!1$ are in the marketed subspace X. Having said this, we advance to the other important assumptions that are made in the CPT.

Assumption 2 (CPT-preferences)

Every agent's utility function can be represented as

$$U^i(\Delta x) = \int_{\mathrm{I\!R}} u^i(\Delta y)\, d\left(T^i \circ \Phi(\Delta y)\right) \quad \textit{for all} \quad \Delta x \in X, \tag{1}$$

where

- u^i *is a two-times differentiable function on* $\mathrm{I\!R} \setminus \{0\}$*, strictly increasing on* $\mathrm{I\!R}$*, strictly concave on* $(0,\infty)$ *and strictly convex on* $(-\infty, 0)$*,*
- T^i *is a differentiable, non-decreasing function from [0,1] onto [0,1] with* $T^i(p) = p$ *for* $p = 0$ *and* $p = 1$ *and with* $T^i(p) > p$ *(*$T^i(p) < p$*) for p small (large),*
- Φ *denotes the cumulative distribution function for the payoffs* Δx*.*

Hence, the utility function u^i captures loss aversion because it needs not be differentiable at 0. Moreover, it is convex-concave. The function T^i transforms the cumulative probabilities as required by Tversky and Kahneman (1992).

The portfolio choice problem is:

$$\max_{\substack{\theta \in \mathrm{I\!R}^{J+1}:\, \sum_{j=0}^{J} q_j\, \theta_j \le w^i \\ x = \sum_{j=0}^{J} A_j\, \theta_j}} U^i\left(x - \beta w^i \mathbb{1}\right) \tag{2}$$

Which can equivalently be written as:

$$\max_{\substack{\theta \in \mathrm{I\!R}^{J+1}:\, \sum_{j=0}^{J} q_j\, \theta_j \le w^i \\ x = \sum_{j=0}^{J} A_j\, \theta_j}} \hat{U}^i\,(x) \tag{3}$$

The CAPM is an equilibrium model. We are therefore interested in analyzing competitive equilibria for the financial market of this paper:

Definition 1

Given a riskfree rate r, a financial market equilibrium consists of a price vector $\overset{*}{q} \in \mathrm{I\!R}^{J+1}$ *with* $\overset{*}{q}_0 = \frac{1}{1+r}$ *and an allocation* $\overset{*}{\theta}{}^i \in \mathrm{I\!R}^{J+1}$, $\mathrm{i} = 1, ..., \mathrm{I}$*, such that*

(i) $\overset{*}{\theta}{}^i$ *maximizes* $U^i(\sum_j A_j \theta^i_j - \beta w^i \mathbb{1})$ *subject to* $\sum_j \overset{*}{q}_j\ \theta^i_j \le\ w^i$, $i = 1, ..., I$*, and*

(ii) $\sum\limits_{i=1}^{I} \overset{*}{\theta}{}^i_j = \bar{\theta}_j$, $j = 1, ..., J$*.*

Note that given the riskfree rate, a financial markets equilibrium determines the J prices of the risky assets by clearing the J markets for the risky assets. Instead of analyzing financial markets equilibria as defined in the Definition 1, in the CAPM it is most useful to first transform the decision problem into some abstract problem that uses the structure of the underlying probability space. To do this, note that a necessary condition for the portfolio decision problem given above to have a solution is that consumers cannot exploit an arbitrage opportunity. Since the CPT utility U^i and hence also the utility $\hat{U}^i$ is strictly increasing, this means that the agent cannot find a portfolio that almost surely delivers positive payoffs without requiring any payments. Asset prices are thus arbitrage free only if the following equation holds:

$$L^2_+ \cap \left\{ x \in L^2(\eta) \,\middle|\, x = \textstyle\sum_{j=0}^J A_j\, \theta_j \text{ where } \sum_{j=0}^J q_j\, \theta_j \leq 0 \right\} = \{0\}\,. \tag{4}$$

Let $q \in \mathrm{I\!R}^{\mathrm{J}+1}$ be an abitrage free price vector. Under Assumption 1, an arbitrage free price vector q needs to satisfy $q_0 > 0$. By the Dalang-Morton-Willinger Theorem (see for example Delbaen 1999), there exists a probability measure π on $(M, \mathcal{M})$, $\pi \sim \eta$ such that $\frac{q_j}{q_0} = I\!E_\pi[A_j]$ for all $j = 1, \ldots, J$. Here we consider discounted prices $\frac{q_j}{q_0}$. Note that $q_0 = \frac{1}{1+r}$. We obtain $q_j = \frac{1}{1+r}\, I\!E_\pi[A_j]$. We can rewrite the pricing rule by defining the Radon-Nikodym Derivative of π with respect to η, the so called likelihood ratio process $\ell = \frac{d\pi}{d\eta} \in L^2(\eta)$ and we obtain $q_j = \frac{1}{1+r}\,\mu(\ell\, A_j) = \frac{1}{1+r}\,\ell \cdot A_j$. At an equilibrium the price system ℓ is also called 'ideal security' (Magill and Quinzii 1996) or 'pricing portfolio' (Duffie 1988). Applying the pricing rule to the portfolio decision problem recognizing the way x is generated by θ, delivers the so called no-arbitrage decision problem

$$\max_{x \in X} \hat{U}^i(x)\,,\; \ell \cdot x \leq (1+r) w^i. \tag{5}$$

To gain intuition on the Dalang-Merton-Willinger Theorem, we briefly consider the case for M finite, $\mathcal{M} = 2^{|M|}$ and $\eta(m) > 0$ for all $m \in M$. The arbitrage free equation (4) implies that

$$\left\{ x = \sum_{j=0}^J A_j\, \theta_j \,\middle|\, \sum_{j=0}^J q_j\, \theta_j \leq 0 \right\} \cap \left\{ h \in L^2(\eta) \,\middle|\, h \geq 0, \sum_{m \in M} h(m) = 1 \right\} = \emptyset.$$

Let $\mathcal{K} = \left\{ x = \sum_{j=0}^J A_j\, \theta_j \,\middle|\, \sum_{j=0}^J q_j\, \theta_j \leq 0 \right\}$. $\mathcal{K}$ defines a sub-space of $L^2(\eta)$. Let $\mathcal{P} = \{h \in L^2(\eta)\, | h \geq 0, \sum_{m \in M} h(m) = 1\}$. Since $\mathcal{K} \cap \mathcal{P} = \emptyset$, then by Farka's Lemma we find a linear functional Ψ on $L^2(\eta)$ with $\Psi(f) = 0$ for $f \in \mathcal{K}$ and $\Psi(h) > 0$ for $h \in \mathcal{P}$. Moreover, by the Riesz Representation Theorem (see Duffie 1988, Chapter I.6) we find $\psi \in L^2(\eta)$ with $\Psi(g) = \mu(\psi\, g)$ for all $g \in L^2(\eta)$. Let $m \in M$ and define h_m by $h_m(m') = 1$ if $m' = m$ and $h_m(m') = 0$ else. h_m is the Arrow security for state m. Obviously $h_m \in \mathcal{P}$ and $0 < \Psi(h_m) = \psi(m)\eta(m)$ for all $m \in M$. Since $\eta(m) > 0$ then $\psi(m) > 0$. We define $\ell = \frac{\psi}{\mu(\psi)}$ and a probability measure π on $(M, \mathcal{M})$ by $\pi(m) = \ell(m)\eta(m)$. We have

$$\mu(\psi)^{-1}\Psi(g) = \sum_{m \in M} g(m)\,\pi(m) = I\!E_\pi\,[g]\,.$$

Consider the following investment: Borrow $\theta_0 = -1$ units of the riskfree asset, to finance $\theta_i = \frac{q_0}{q_i}$ units of asset $i \in \{1, \ldots, J\}$ ($\theta_k = 0$ for $k \neq 0, i$). Then, $x = \sum_{j=0}^{J} \theta_j\, A_j \in \mathcal{K}$ and thus $\mu(\psi)^{-1}\Psi(x) = I\!E_\pi\,[x] = 0$. It follows:

$$q_j = \frac{q_0}{I\!E_\pi\,[1\!\!1]} I\!E_\pi\,[A_j] = \frac{1}{1+r} I\!E_\pi\,[A_j]\,.$$

Since we restrict the pricing rule just described to X, we can assume without loss of generality[2] that $\ell \in X$. In fact, if $\ell \notin X$, we can decompose ℓ into one part $\ell_\parallel$ in X and one part $\ell_\perp$ orthogonal to X. Since for all $x \in X$, $\ell_\perp \cdot x = 0$, the pricing rule can be rewritten as $\ell_\parallel \cdot x$. Thus, we assume $\ell \in X$. Back to the no-arbitrage decision problem (5), we can now give an equivalent definition of financial markets equilibria that is easier to analyze than the Definition 1:

Definition 2

Given a riskfree rate r, a financial market equilibrium *consists of a price vector $\overset{*}{\ell} \in X$ and an allocation $\overset{*}{x}{}^i \in L^2(\eta)$, $i = 1, ..., I$, such that (i) $\overset{*}{x}{}^i$ maximizes $\hat{U}^i(x)$ subject to $x \in X$ and $\overset{*}{\ell} \cdot x \leq\ (1+r)\, w^i$, $i = 1, ..., I$, and (ii) $\sum_{i=1}^{I} \overset{*}{x}{}^i_\perp = \omega$.*

3 Results

Before we exploit the specific assumptions, Assumption 1 and 2, made in the previous section we will briefly recall what can already be said with respect to asset prices.

Proposition 1 (Asset Pricing)

Let y be any payoff in X and define $q(y) := \sum_j q_j\theta_j$ for some θ with $y = \sum_j A_j\theta_j$. Then we obtain that in any financial market equilibrium the likelihood ratio process ℓ is the only risk factor of the model, i.e. $q(y) = \frac{1}{1+r}\,(\mu(y) + cov(y, \ell))$.

[2]This assumption just refers to the pricing rule $\ell \cdot x$ and not to the way ℓ is obtained. It might occur that the new ℓ cannot be written as Radon-Nikodym Derivative with respect to some equivalent probability measure.

Proof.
This pricing formula follows immediately from No-arbitrage pricing:

$$(1+r)\, q_j = \ell \cdot A_j = \mu(\ell A_j) = \mu(\ell)\mu(A_j) + cov(\ell, A_j)$$

noting that $\mu(\ell) = 1$.

$\square$

We now demonstrate that, given returns are normally distributed (Assumption 1), then utility functions according to CPT (Assumption 2) are actually functions of the mean and variance only.

Proposition 2 (Mean-Variance Preferences)

With normally distributed returns (Assumption 1), preferences according to CPT (Assumption 2) are mean-variance preferences that are increasing in mean, i.e. for all $\Delta x \in X$

$$U^i(\Delta x) = V^i(\mu(\Delta x), \sigma(\Delta x))$$

and V^i is strictly increasing in μ.

Proof.
For any portfolio θ let $\mu_\theta = \mu(\Delta x_\theta) = \mu(\sum_j A_j\theta_j - \ell \cdot x_\theta)$, $\sigma_\theta = \sigma(\Delta x_\theta) = \sigma(\sum_j A_j\theta_j)$, denote the resulting mean-variances of the portfolio's relative payoff. By Assumption 1 each individual asset payoff is normally distributed with parameters μ_j, σ_j hence also the portfolio's relative payoff is normally distributed with the parameters $\mu_\theta, \sigma_\theta$. That is to say, applying equation (1), the agent evaluates changes in wealth according to

$$U^i(\Delta x) = \int_{\mathbb{R}} u^i(\Delta y)\, d\,(\, T^i \circ \Phi_{\mu_\theta,\sigma_\theta}(\Delta y)) \quad \text{for all} \quad \Delta x \in X.$$

Let $\hat{\Phi}(x) := \Phi(\frac{x-\mu}{\sigma})$ denote the standardized cumulative normal distribution. Then using the transformation of variables $\Delta y \to \sigma_\theta \Delta y + \mu_\theta$ we obtain that

$$U^i(\Delta x) = \int_{\mathbb{R}} u^i(\Delta y\, \sigma_\theta + \mu_\theta)\, d\,(\, T^i \circ \hat{\Phi}(\Delta y)) \quad \text{for all} \quad \Delta x \in X.$$

Hence U^i is a function of the portfolio's mean and variances. Moreover, since u^i is strictly increasing and since T^i is non-decreasing with $T^i(p) = p$ for $p = 0, 1$, U^i is strictly increasing in the mean. The same properties carry over to the function $\hat{U}^i$, since it is identical to U^i up to the shift in the mean $\mu_\theta + (1+r)\, w^i$. $\square$

Note that for proving this last result, we essentially use that u^i is strictly increasing and that T^i is non-decreasing. Thus, Proposition 2 applies to all utility functions satisfying equation (1) where u^i is strictly increasing and T^i non-decreasing with $T^i(p) = p$ for $p = 0, 1$.

The mean-variance property of preferences is the main property to derive the Tobin Separation Principle and thus to derive the Mutual Fund Theorem:

Proposition 3 (Tobin Separation)

Let $x^i \in \arg\max_{x\in X} \hat{U}^i(x)$ s.t. $\ell \cdot x \leq (1+r)\, w^i$ for $i = 1, \ldots, I$ and suppose that $\mu(\omega) > \ell \cdot \omega$. Then $x^i = \tilde{\psi}^i \mathbb{1} - \tilde{\phi}^i \ell$ for some scalars $\tilde{\phi}^i \geq 0, \tilde{\psi}^i$ for every $i = 1, ..., I$.

Proof.

We prove Proposition 3 by the following four steps.

(1) *Budget restriction holds with equality.*

Let $x^i \in \arg\max_{x\in X} \hat{U}^i(x)$ s.t. $\ell \cdot x \leq (1+r)\, w^i$. Since there is a riskless asset and since the function $\hat{U}^i$ is increasing in μ, then at any optimal solution x^i the budget restriction holds with equality, i.e. $\ell \cdot x^i = (1+r)\, w^i$ for all i. Otherwise it would be possible to further increase the utility by buying the riskfree asset.

(2) *Agents are variance averse.*

Let $x^i \in \arg\max_{x\in X} \hat{U}^i(x)$ s.t. $\ell \cdot x \leq (1+r)\, w^i$ and suppose that $\mu(\omega) > \ell \cdot \omega$. Since $\mu(\omega) > \ell \cdot \omega$, then $\mu(\frac{\omega - q(\omega)}{q(\omega)}) > r$, i.e. the return on the market portfolio is greater than the risk free return r and thus, if investor i were not variance averse at x^i, she would then short sell the risk free asset and buy the market portfolio, increasing in this way her utility and contradicting the optimality of x^i.

(3) *Tobin's Separation.*

Let $x^i \in \arg\max_{x\in X} \hat{U}^i(x)$ s.t. $\ell \cdot x \leq (1+r)\, w^i$ and suppose that $\mu(\omega) > \ell \cdot \omega$. Decompose $x^i = y^i + z^i$ where z^i is perpendicular to $\mathbb{1}$ and ℓ and $y_i \in \text{span}(\mathbb{1}, \ell) \subset$ X. From the decomposition it follows that $\ell \cdot z^i = 0$ so that $y^i \in X$ is budget feasible. Moreover, from z^i being perpendicular to $\mathbb{1}$ it is obtained that $\mu(x^i) = \mu(y^i)$. Suppose that $z^i \neq 0$, then

$$\begin{aligned}
\sigma^2(x^i) &= \sigma^2(x^i_\perp) = \|x^i_\perp\|^2 = \|y^i_\perp + z^i_\perp\|^2 = \|y^i_\perp\|^2 + \|z^i_\perp\|^2 \\
&> \|y^i_\perp\| + \|z^i_\perp\|^2 = \sigma^2(y^i_\perp) = \sigma^2(y^i)
\end{aligned}$$

because $z^i_\perp$ is perpendicular to $y^i_\perp$, where the subscript $\perp$ denotes the component of each vector orthogonal to $\mathbb{1}$. Since by (2) investors are variance averse at the optimal allocation x^i, then $z^i = 0$. Therefore $x^i = \tilde{\psi}^i \mathbb{1} - \tilde{\phi}^i \ell$ for some scalars $\tilde{\phi}^i, \tilde{\psi}^i$ for every $i = 1, ..., I$, i.e. Tobin's Separation holds.

(4) $\tilde{\phi}^i \geq 0$.
From (3) we obtain $x^i = \tilde{\psi}^i \mathbb{1} - \tilde{\phi}^{\mathrm{i}} \ell$. It remains to show that $\tilde{\phi}^i \geq 0$. From $x^i = \tilde{\psi}^i \mathbb{1} - \tilde{\phi}^{\mathrm{i}} \ell$ and the budget equality $\ell \cdot x^i = (1+r)\, w^i$, it follows $\mu(x^i) = (1+r)\, w^i + \tilde{\phi}^i \sigma^2(\ell)$ and $\sigma(x^i) = |\tilde{\phi}^i|\, \sigma(l)$. If $\tilde{\phi}^i < 0$, then, because by Proposition 2 V^i is increasing in μ one could increase the utility by buying the asset $\tilde{\psi}^i \mathbb{1} + \tilde{\phi}^{\mathrm{i}} \ell$, a contradiction to the optimality of x^i.

□

The next result shows that without loss of generality we can work in the famous mean-variance diagram.

Corollary 1 (Mean-Variance Principle)

Suppose that $\mu(\omega) > \ell \cdot \omega$, then the decision problems

$$x^i \in \arg\max_{x \in X} \hat{U}^i(x) \text{ s.t. } \ell \cdot x \leq (1+r)\, w^i$$

and

$$(\mu^i, \sigma^i) \in \arg \max_{(\mu,\sigma) \in \mathbb{R} \times \mathbb{R}_+} V^i(\mu - (1+r)\, w^i, \sigma) \text{ s.t. } \mu - q\sigma = (1+r)\, w^i$$

are equivalent, where $q = \sigma(\ell) > 0$.

Proof.
From step (4) of the proof of Proposition 3 we obtain $\mu(x^i) = (1+r)\, w^i + \sigma(x^i)\, \sigma(\ell)$ for any optimal solution x^i of the first decision problem. Moreover, by Proposition 2, $(\mu(x^i), \sigma(x^i))$ maximizes V^i. On the other hand, for any solution (μ^i, σ^i) of the second decision problem we can find unique $\tilde{\psi}^i$ and $\tilde{\Phi}^i \geq 0$ such that $x^i = \tilde{\psi}^i \mathbb{1} - \tilde{\phi}^{\mathrm{i}} \ell$ has mean μ^i and variance σ^i. ¿From the budget restriction of the second decision problem, it follows that x^i is budget feasible for investor i. Moreover, by Proposition 2, x^i maximizes $\hat{U}^i$.

□

Now we are in a position to consider the equilibrium consequences of what we discovered so far:

Proposition 4 (Mutual Fund Theorem)

Given a riskfree rate r, let $(\overset{*}{\ell}, \overset{*}{x})$ with $\overset{*}{x} = (\overset{*}{x}{}^1, ..., \overset{*}{x}{}^I)$ be a financial market equilibrium and suppose $\mu(\omega) > \overset{*}{\ell} \cdot \omega$. Then there exist scalars $\phi^i \in \mathbb{R}_+$, $i = 0, 1, ..., I$, and scalars $\psi^i \in \mathbb{R}$, $i = 0, 1, ..., I$ such that $\overset{*}{\ell} = \psi^0 \mathbb{1} + \phi^0 \omega$ and $\overset{*}{x}{}^i = \psi^i \mathbb{1} - \phi^{\mathrm{i}} \omega$ for every $i = 1, ..., I$.

Proof.
By Tobin's Separation (Proposition 3) $\overset{*}{x}^i = \tilde{\psi}^i \mathbb{1} - \tilde{\phi}^{\mathrm{i}} \overset{*}{\ell}$ for some scalars $\tilde{\phi}^i \geq 0, \tilde{\psi}^i$ for every $i = 1, ..., I$. Since in equilibrium $\sum_i \overset{*}{x}^i_\perp = \omega_\perp$ we get $\sum_i \overset{*}{x}^i = \alpha \mathbb{1} + \omega$ for some $\alpha \in \mathbb{R}$. Hence there exist scalars $\phi^0 \geq 0, \psi^0$ such that $\overset{*}{\ell} = \psi^0 \mathbb{1} - \phi^0 \omega$. Using this last equation, we obtain $\overset{*}{x}^i = \psi^i \mathbb{1} - \phi^{\mathrm{i}} \omega$ for some scalars $\phi^i \geq 0, \psi^i$ for every $i = 1, ..., I$.

□

The main conclusion of the CAPM, the Security Market Line Theorem, is now straightforward:

Proposition 5 (Security Market Line Theorem)
Suppose that the gross return of the market portfolio is greater than the risk free gross return, i.e. $\mu(\omega) > \overset{*}{\ell} \cdot \omega$. *Then, given asset payoffs are normally distributed (Assumption 1) and agents have prospect theory utility functions (Assumption 2), at every financial market equilibrium* $(\overset{*}{\ell}, \overset{*}{x})$ *the security market line holds, i.e. for any payoff* $y \in X$

$$\mu(r_y) - r = \frac{cov(r_y, r_\omega)}{\sigma^2(r_\omega)} (\mu(r_\omega) - r) \tag{6}$$

where $r_y = \frac{y - q(y)}{q(y)}$ *and* $r_\omega = \frac{\omega - q(\omega)}{q(\omega)}$.

Proof.
Inserting the Mutual Fund Theorem expression for $\overset{*}{\ell} = \psi^0 \mathbb{1} - \phi^0 \omega$ into the asset pricing formula derived in Proposition 1, gives:

$$(1 + r) q(y) = \mu(y) - \phi^0 cov(\omega, y).$$

Applying this formula for $y = \omega$ we can determine $\phi^0 > 0$ as

$$\phi^0 = \frac{\mu(\omega) - (1 + r)\, q(\omega)}{\sigma^2(\omega)}.$$

Hence we obtain:

$$(1 + r) q(y) = \mu(y) - \frac{\mu(\omega) - (1 + r) q(\omega)}{\sigma^2(\omega)}\; cov(\omega, y),$$

which can also be written as

$$(1 + r)\, q(y) - \mu(y) = \frac{cov(\omega, y)}{\sigma^2(\omega)} \left((1 + r) q(\omega) - \mu(\omega)\right).$$

Dividing this equation by $q(y)$ and dividing the numerator and the denominator on the RHS of this equation by $q^2(\omega)$ delivers the result.

$\square$

So far we have assumed that the market portfolio has a higher return than the risk free rate. As the following remark shows, for the utility functions of Kahneman and Tversky (1992), this has indeed to be the case at any financial market equilibrium. As we show later this will also be the case for the utility function that we propose.

Remark (The condition $\mu(\omega) > \overset{*}{\ell} \cdot\omega$ needs to hold in equilibrium)

Suppose $\mu(\omega) \le \overset{*}{\ell} \cdot\omega$, then at any financial market equilibrium $(\overset{*}{\ell}, \overset{*}{x})$ we get $\mu(\overset{*}{x}{}^i) = (1+r)\, w^i$ for $i = 1, \dots, I$. In fact, the condition $\mu(\omega) \le \overset{*}{\ell} \cdot\omega$ implies that $\sum_{i=1}^{I} \mu(\overset{*}{x}{}^i) \le (1+r) \sum_{i=1}^{I} w^i$ and thus there exists at least one investor $j \in \{1, \dots, I\}$ such that $\mu(\overset{*}{x}{}^j) \le (1+r)\, w^j$. Note that assets $\alpha\, \overset{*}{x}{}^j + (1-\alpha)(1+r)\, w^j$ are budget feasible for investor j, for all $\alpha \in \mathbb{R}$ and moreover,

$$
\begin{aligned}
\mu\left(\alpha\, \overset{*}{x}{}^j + (1-\alpha)(1+r)\, w^j\right) &= \alpha\left(\mu(\overset{*}{x}{}^j) - (1+r)\, w^i\right) + (1+r)\, w^j, \\
\sigma\left(\alpha\, \overset{*}{x}{}^j + (1-\alpha)(1+\overset{*}{r})\, w^j\right) &= |\alpha|\, \sigma(\overset{*}{x}{}^j)
\end{aligned}
$$

for all $\alpha \in \mathbb{R}$. Take $\alpha = -1$, then from the previous equations for mean and variance, it follows that, if $\mu(\overset{*}{x}{}^j) < (1+\overset{*}{r}) w^j$, then investor j could increase the mean of her portfolio, without affecting the variance and thus could increase her utility (strictly increasing in μ) by short selling $\overset{*}{x}{}^j$ and buying the risk free asset. This would contradict the optimality of $\overset{*}{x}{}^j$, thus $\mu(\overset{*}{x}{}^j) = (1+r)\, w^j$. $\sum_{i=1}^{I} \mu(\overset{*}{x}{}^i) \le (1+r) \sum_{i=1}^{I} w^i$ together with the equality $\mu(\overset{*}{x}{}^j) = (1+r) w^j$ which holds for the selected investor j, imply that $\mu(\overset{*}{x}{}^i) = (1+r) w^i$ for all $i \in \{1, \dots, I\}$. Since $\sigma(\omega) > 0$, we find $k \in \{1, \dots, I\}$ with $\sigma^k = \sigma(\overset{*}{x}{}^k) > 0$, i.e. investor's k optimal choice $\overset{*}{x}{}^k$ is Gaussian distributed with mean $(1+r)\, w^k$ and strictly positive variance. Thus, $V^k(0, \sigma^k) > V^k(0, 0)$, where V^k is the mean-variance preference induced by U^k (Proposition 2). But this contradict the CPT assumption for U^k (see Introduction and Assumption 2), since in fact no investor who dislikes losses by a factor 2.25 as compared to gains, and chooses the risk free investment as reference point, prefers payoff $\overset{*}{x}{}^k$ to the risk free investment! In particular, let us consider the utility index u and transformation T proposed by Tversky and Kahneman (1992):

$$u(x) = \begin{cases} x^\alpha & \text{for } x \geq 0, \\ -\lambda(-x)^\beta & \text{for } x < 0, \end{cases} \tag{7}$$

$$T(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}}. \tag{8}$$

where $\beta = \alpha = 0.88$, $\lambda = 2.25$ and $\gamma = 0.61$ for gains and $\gamma = 0.69$ for losses. The induced mean-variance preference V^k satisfies $V^k(0, \sigma) = c\,\sigma^\alpha$ for all $\sigma \geq 0$, where $c \approx -0.34$. Therefore $V^k(0, \sigma) < V^k(0, 0)$ for all $\sigma > 0$, i.e. no investor will choose a positive variance.
In summary, this remark has shown that the existence of financial market equilibria for the case $\mu(\omega) \leq \ell \cdot \omega$ is not consistent with Tversky and Kahneman (1992) assumptions for the utility functions.

4 Existence of Equilibria

Up to now we have assumed that a financial market equilibrium exists. This assumption implies that each investor's decision problem has a solution (Definition 1 and 2). The no arbitrage condition expressed by equation (4), is a necessary condition for the existence of a solution of the investor's decision problem, but it is not sufficient. To understand why the investor‘s decision problem may fail to have a solution, let us consider the following general example.

Suppose Assumptions 1 and 2 are satisfied and the no arbitrage condition (4) holds. Let ℓ be the pricing portfolio. We suppose, as in Proposition 3, 4 and 5, that $\mu(\omega) > \ell \cdot \omega$ and we define $\delta(\omega) = \frac{\mu(\omega) - \ell \cdot \omega}{\sigma(\omega)}$. Note that $\delta(\omega)$ corresponds the the Sharpe Ratio of the market portfolio $\frac{\mu(r_\omega) - r}{\sigma(r_\omega)}$ (Sharpe 1994). Let $\tilde{\omega} = (1+r)\,\frac{w^k}{\ell \cdot \omega}\,\omega$. Then $\tilde{\omega}$ is budget feasible for investor k and $\delta(\tilde{\omega}) = \delta(\omega) > 0$. We consider the leveraged portfolio $x(\alpha) = \alpha\,\tilde{\omega} + (1-\alpha)(\ell \cdot \tilde{\omega})\,\mathbb{1}$. Then

$$\begin{aligned} \ell \cdot x(\alpha) &= \ell \cdot \tilde{\omega}, \\ \mu\,(x(\alpha)) &= \alpha\,(\mu(\tilde{\omega}) - \ell \cdot \tilde{\omega}) + \ell \cdot \tilde{\omega}, \end{aligned}$$

i.e. $x(\alpha)$ is budget feasible and $\mu\,(x(\alpha)) \nearrow \infty$ as $\alpha \nearrow \infty$.

Suppose, for sake of simplicity, that investor k does not transform the distribution function, i.e. $T^k(p) = p$ for all $p \in [0, 1]$ and uses the utility index

$$u^k(x) = \begin{cases} f(x) & \text{for } x > 0; \\ 0 & \text{for } x = 0; \\ -\lambda\, f(-x) & \text{for } x < 0, \end{cases}$$

where $f(x) = x$ is the linear function on $(0, \infty)$ and $\lambda \approx 2.25$ (see Barberis, Huang and Santos 2001). We consider the utility function U^k on the set $\{x(\alpha) \mid \alpha > 0\}$. We have:

$$\begin{aligned}
V^k(\alpha) &= U^k(x(\alpha)) = V^k(\alpha\, \mu(\tilde{\omega}) - \alpha\, \ell \cdot \tilde{\omega}, \alpha\sigma(\tilde{\omega})) \\
&= \int_{\mathbb{R}} u^k \left(\alpha(\sigma(\tilde{\omega})x + \mu(\tilde{\omega}) - \ell \cdot \tilde{\omega})\right) d\hat{\Phi}(x) \\
&= \alpha \Bigg[\int_{-\delta(\omega)}^{\infty} \left(\sigma(\tilde{\omega})\, x + \mu(\tilde{\omega}) - \ell \cdot \tilde{\omega}\right) d\hat{\Phi}(x) \\
&\quad + \int_{-\infty}^{-\delta(\omega)} \lambda \left(\sigma(\tilde{\omega})\, x + \mu(\tilde{\omega}) - \ell \cdot \tilde{\omega}\right) d\hat{\Phi}(x) \Bigg] \\
&= \alpha \left[(\mu(\tilde{\omega}) - \ell \cdot \tilde{\omega}) \left(1 + (\lambda - 1)\, \hat{\Phi}(-\delta(\omega))\right) + \sigma(\tilde{\omega})\, (\lambda - 1)\, \hat{\varphi}(\delta(\omega)) \right] \\
&= \alpha\, c(\lambda, \delta(\omega)).
\end{aligned}$$

where $\hat{\varphi}$ is the probability density of the standard normal distribution. For the U.S. economy, Mehra (2003) gives an estimate of the long-term Sharpe Ratio of $\delta(\omega) = 0.34$ and $c(2.25, 0.34) > 0$. Thus investor k can infinitely increase her utility by infinitely leveraging the market portfolio and therefore no solution to her decision problem would exist.

Tversky and Kahnemann (1992).
We restrict now our attention to the utility index and probability transformation proposed by Tversky and Kahnemann, which are defined in equations (7) and (8). For the reasoning of this subsection it will be important to allow for some possibly small heterogeneity in agents preferences. Therefore let:

$$u^i(x) = \begin{cases} x^{\alpha^i} & \text{for } x \geq 0, \\ -\lambda^i(-x)^{\alpha^i} & \text{for } x < 0, \end{cases} \tag{9}$$

$$T^i(p) = \frac{p^{\gamma^i}}{\left(p^{\gamma^i} + (1-p)^{\gamma^i}\right)^{\frac{1}{\gamma^i}}}. \tag{10}$$

The Corollary 1 to the Tobin's Separation Theorem implies that for any pricing portfolio ℓ with $\mu(\omega) > \ell \cdot \omega$, x^i solves

$$\max_{x \in X} \hat{U}^i(x) \text{ s.t. } \ell \cdot x \leq (1 + r)\, w^i$$

iff $(\mu(x^i), \sigma(x^i))$ solves

$$\max_{(\mu,\sigma) \in \mathbb{R} \times \mathbb{R}_+} V^i(\mu - (1 + r)\, w^i, \sigma) \text{ s.t. } \mu - q\sigma = (1 + r)\, w^i,$$

where $q = \sigma(l) > 0$. We consider $\hat{U}^i(x^i) = V^i(q\sigma, \sigma)$ induced by (7) and (8). We obtain:

$$\begin{aligned} V^i(q\sigma,\sigma) &= \sigma^{\alpha^i}\left[\int_{-q}^{\infty}(x+q)^{\alpha^i}\, d(T^i\circ\hat{\Phi}(x)) - \lambda^i \int_{-\infty}^{-q}(-x-q)^{\alpha^i}\, d(T^i\circ\hat{\Phi}(x))\right] \\ &= \sigma^{\alpha^i} f^i(q), \end{aligned}$$

where $f^i(q) = \left[\int_{-q}^{\infty}(x+q)^{\alpha^i}\, d(T^i\circ\hat{\Phi}(x)) - \lambda^i \int_{-\infty}^{-q}(-x-q)^{\alpha^i}\, d(T^i\circ\hat{\Phi}(x))\right]$ is continuous and strictly increasing on $\mathbb{R}_+$, $f^i(0) \approx -0.34$ and $f^i(q) \nearrow \infty$ as $q \nearrow \infty$. Thus for all agents $i = 1, \ldots, I$, there exists exactly one $q^i > 0$ such that $f^i(q^i) = 0$. It follows that for $0 < q < q^i$, investor i's optimal allocation is the risk free asset, for $q = q^i$ the investor is indifferent between all possible allocations and for $q > q^i$ investor i's optimal behavior consists in infinitely leveraging the market portfolio. Thus as soon as the investors are a little heterogenous no equilibrium exists. Figure 2 shows the Tversky and Kahnemann indifference curves in the (σ, μ) plane. For all pricing portfolios ℓ such that $q \neq q^i$, no solution to the individual optimization problem can exists under Tversky and Kahneman's (1992) assumptions for the utility functions.

Our proposal.
We suggest to consider the following utility index

$$u(x) = \begin{cases} -\lambda^+ \exp(-\alpha\, x) + \lambda^+ & \text{for } x \geq 0, \\ \lambda^- \exp(\alpha\, x) - \lambda^- & \text{for } x < 0, \end{cases} \tag{11}$$

where $\alpha \in (0,1)$, $\lambda^- > \lambda^+ > 0$. Figure 1 shows that $u(x)$ approximates the Tversky and Kahnemann (1992) utility index very well for values around zero. We presume that the experimental evidence given for the utility specification of Kahneman and Tversky (1979) foremost concerns the shape of the utility function around zero. Note also that the utility function we propose is different to that of Kahneman and Tversky (1979) for very high stakes because it is less linear than theirs. Our theoretical analysis thus suggest to conduct further experiments in this direction. For sake of simplicity we take $T(p) = p$ for all $p \in [0,1]$. As it will be seen later for this specification financial markets equilibria are robust with respect to introducing some heterogeneity of agents' preferences. For any agent at any consumption bundle, $x \in X$, $\mu(\Delta x) = \mu$, $\sigma(\Delta x) = \sigma$ we obtain (see Appendix):

$$\begin{aligned} U(\Delta x) &= V(\mu,\sigma) = \int_{\mathbb{R}} u(\sigma\Delta y + \mu)\, d\hat{\Phi}(\Delta y) \\ &= (\lambda^+ + \lambda^-)\hat{\Phi}\left(\frac{\mu}{\sigma}\right) - \lambda^- \\ &\quad + e^{\frac{1}{2}\sigma^2\alpha^2}\left[\lambda^- e^{\alpha\mu}\,\hat{\Phi}\left(-\frac{\mu}{\sigma} - \alpha\sigma\right) - \lambda^+ e^{-\alpha\mu}\,\hat{\Phi}\left(\frac{\mu}{\sigma} - \alpha\,\sigma\right)\right]. \end{aligned}$$

Figure 3 shows the indifference curves of V in the mean and standard deviation space. The partial derivatives of V are

$$\begin{aligned}\partial_\mu V(\mu,\sigma) &= \alpha\, e^{\frac{1}{2}\alpha^2\sigma^2}\left[\lambda^-\, e^{\alpha\mu}\,\hat{\Phi}\left(-\frac{\mu}{\sigma}-\alpha\sigma\right)+\lambda^+\, e^{-\alpha\mu}\,\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)\right],\\ \partial_\sigma V(\mu,\sigma) &= \alpha^2\,\sigma\; e^{\frac{1}{2}\alpha^2\sigma^2}\left[\lambda^-\, e^{\alpha\mu}\,\hat{\Phi}\left(-\frac{\mu}{\sigma}-\alpha\sigma\right)-\lambda^+\, e^{-\alpha\mu}\,\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)\right]\\ &\quad -\alpha\,(\lambda^- - \lambda^+)\,\hat{\varphi}\left(\frac{\mu}{\sigma}\right).\end{aligned}$$

where $\hat{\varphi} = \hat{\Phi}'$ is the density function for the standard normal distribution. The ratio

$$S(\mu,\sigma) = -\frac{\partial_\sigma V(\mu,\sigma)}{\partial_\mu V(\mu,\sigma)}$$

gives the slope of the indifference curve at some point in the mean and standard deviation space. The following properties hold:

(i) $\partial_\mu V(\mu,\sigma) > 0$,

(ii) $\partial_\sigma V(\mu,0) = 0$, $\partial_\sigma V(\mu,\sigma) < 0$ for $\sigma > 0$ [3]
and $\lim_{\sigma\to\infty}\partial_\sigma V(\mu,\sigma) = 0$ for all $\mu > 0$,

(iii) $S(\mu,0) = 0$ and $S(\mu,\sigma) > 0$ for all $\mu > 0$,

(iv) $\lim_{\frac{\mu}{\sigma}\to\infty} S(\mu,\sigma) = \alpha\,\sigma$ for all $\sigma > 0$ fix,

(v) $\lim_{\sigma\to\infty} S(\mu,\sigma) = \infty$ for all $\mu > 0$.

The proof is given in the Appendix. Note that property (ii) implies that the condition $\mu(\omega) > \overset{*}{\ell}\cdot\omega$ needs to hold in equilibrium also under our proposal for the utility index. In fact, as we have already shown in the Remark in the previous section, $\mu(\omega) \leq \overset{*}{\ell}\cdot\omega$ holds in equilibrium only if some investor's optimal allocation is a risky investment with same return as the risk-free asset, a contradiction to property (ii).
The final property we need to show is the quasi-concavity of V. Tobin (1958) has

[3]Property 2 expressed in Meyer (1987) states that when the class of considered risks is generated by a location and scale parameter condition, then for V given by (1), $\partial_\sigma V(\mu,\sigma) \leq 0$ if and only if the utility index u satisfies $u''(\mu+\sigma\,x) \leq 0$ for all $\mu+\sigma\,x$. Since the utility index (11) does not satisfy $u''(\mu+\sigma\,x) \leq 0$ for all $\mu+\sigma\,x$, one could expect our statement to contradict Property 2 in Meyer (1987). But this is not the case, since the necessity of the condition on u for Property 2 in Meyer (1987) holds, if and only if all considered risks have bounded support, which is obviously not the case under the normal distribution assumption. Indeed, our example shows that the necessity condition does not hold for risks with unbounded support.

already pointed out that quasi-concavity of the mean-variance utility function V is ultimately linked to the concavity of the von Neumann-Morgenstern utility function u. Indeed, as Sinn (1983) has shown, concavity of u easily implies quasi-concavity of V. In the case of prospect theory u is however convex-concave. Hence, quasi-concavity of V depends on whether on average the distribution of Δx puts more weight on the convex or on the concave part of u. The condition $\mu(\omega) > \overset{*}{\ell} \cdot \omega$, which needs to hold in equilibrium as shown above, ensures that equilibrium allocation on average have positive excess return. Note also that loss-aversion, i.e. the feature that u is steeper for losses than for gains, contributes to the concavity of u and hence to the quasi-concavity of V. Indeed, it turns out that our choice of the parameters λ^+, λ^- and α leads to a quasi-concave utility function V (see Figure 3). This puts us into the position of proving our final claim:

Proposition 5 (Existence of CAPM-equilibria)

Under the assumptions (1) and (2) and for the specification of the CPT-utility functions given by (9), for any given riskfree rate r, there exist financial market equilibria with $\mu(\omega) > \overset{*}{\ell} \cdot \omega$.

Proof.

Consider the standard deviation of the market portfolio, $\sigma(\omega)$. by the mean-variance-principle (Corollary 1), we need to find a price $\overset{*}{q}$ such that $\sum_i \sigma^i(\overset{*}{q}) = \sigma(\omega)$, where

$$\sigma^i \in \arg\max_{\sigma \in \mathbb{R}_+} V^i(q\sigma, \sigma).$$

From the boundary behavior of the agents indifference curves (i) to (v) that we showed above, it follows for all $i = 1, \ldots, I$ that for $q \to 0 \quad \sigma^i(q) = 0$ and for $q \to \infty \quad \sigma^i(q) = \infty$. Hence for sufficiently small prices of risk, q, market demand $\sum_i \sigma^i(\overset{*}{q})$ is smaller than market supply $\sigma(\omega)$ while for sufficiently large prices it is larger. Since by the quasi-concavity of preferences demand is continuous, from the intermediate value theorem we get the existence of some equilibrium with $\overset{*}{q} > 0$. Finally, note that $\overset{*}{q} > 0$ is equivalent to $\mu(\omega) > \overset{*}{\ell} \cdot \omega$.

□

5 Conclusion

Under the assumption of normally distributed returns, we have shown that the Cumulative Prospect Theory of Tversky and Kahneman (1992) is consistent with the Capital Asset Pricing Model in the sense that in every financial market equilibrium the Security Market Line Theorem holds. However, we did also show that under the specific functional forms suggested by Tversky and Kahneman (1992) financial market equilibria do not exist. We suggested an alternative functional form consistent with the results of Tversky and Kahneman for which equilibria do exist.

The functional form we suggest differs from that of Kahneman and Tversky (1979) with respect to the behavior for large stakes. This suggests to collect experimental evidence with large stakes since for the existence of equilibria the behavior of agents at the boundary of their consumption space is essential.

The CAPM analyzed in this paper is a standard two periods model. While much of the intuition for the general intertemporal CAPM can already be given in the two periods model still it would be very interesting to check the consistency of prospect theory and the CAPM also in the intertemporal model. In particular it is unclear whether the intertemporal CAPM is consistent with possible shifts in the reference point. Moreover, in the intertemporal model returns will be endogenous and most likely not normally distributed, giving prospect theory some chance to differ from mean-variance analysis. For recent papers incorporating some aspects of prospect theory in intertemporal models see Benartzi and Thaler (1995) and also Barberis, Huang and Santos (2001).

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Appendix

First we prove that

(0)

$$
\begin{aligned}
V(\mu,\sigma) &= (\lambda^+ + \lambda^-)\hat{\Phi}\left(\frac{\mu}{\sigma}\right) - \lambda^- \\
&\quad + e^{\frac{1}{2}\sigma^2\alpha^2}\left[\lambda^-\, e^{\alpha\mu}\,\hat{\Phi}\left(-\frac{\mu}{\sigma} - \alpha\sigma\right) - \lambda^+\, e^{-\alpha\mu}\,\hat{\Phi}\left(\frac{\mu}{\sigma} - \alpha\,\sigma\right)\right],
\end{aligned}
$$

if

$$
u(x) = \begin{cases} -\lambda^+ \exp(-\alpha\, x) + \lambda^+ & \text{for } x \geq 0, \\ \lambda^- \exp(\alpha\, x) - \lambda^- & \text{for } x < 0, \end{cases} \tag{12}
$$

where $\alpha \in (0,1)$, $\lambda^- > \lambda^+ > 0$.

(i) $\partial_\mu V(\mu,\sigma) > 0$,

(ii) $\partial_\sigma V(\mu,0) = 0$, $\partial_\sigma V(\mu,\sigma) < 0$ for $\sigma > 0$ and $\lim_{\sigma\to\infty} \partial_\sigma V(\mu,\sigma) = 0$ for all $\mu > 0$,

(iii) $S(\mu,0) = 0$ and $S(\mu,\sigma) > 0$ for all $\mu > 0$,

(iv) $\lim_{\frac{\mu}{\sigma}\to\infty} S(\mu,\sigma) = \alpha\,\sigma$ for all $\sigma > 0$ fix,

(v) $\lim_{\sigma\to\infty} S(\mu,\sigma) = \infty$ for all $\mu > 0$.

Proof.
(0)

$$
\begin{aligned}
U(\Delta x) &= V(\mu,\sigma) = \int_{\mathbb{R}} u(\sigma\Delta y + \mu)\, d\hat{\Phi}(\Delta y) \\
&= \int_{-\frac{\mu}{\sigma}}^{\infty} \left(-\lambda^+ e^{-\alpha(\sigma x+\mu)} + \lambda^+\right) d\hat{\Phi}(x) + \int_{-\infty}^{-\frac{\mu}{\sigma}} \left(\lambda^- e^{\alpha(\sigma x+\mu)} - \lambda^-\right) d\hat{\Phi}(x) \\
&= \lambda^+ \left(1 - \hat{\Phi}\left(-\frac{\mu}{\sigma}\right)\right) - \lambda^-\,\hat{\Phi}\left(-\frac{\mu}{\sigma}\right) + \\
&\quad + \lambda^-\, e^{\alpha\,\mu} \int_{-\infty}^{-\frac{\mu}{\sigma}} e^{\alpha\sigma x}\, d\hat{\Phi}(x) - \lambda^+\, e^{-\alpha\,\mu} \int_{-\frac{\mu}{\sigma}}^{\infty} e^{-\alpha\sigma x}\, d\hat{\Phi}(x) \\
&= (\lambda^+ - \lambda^-)\,\hat{\Phi}\left(\frac{\mu}{\sigma}\right) - \lambda^- + \\
&\quad + \lambda^-\, e^{\alpha\,\mu} \int_{\frac{\mu}{\sigma}}^{\infty} e^{-\alpha\sigma x}\, d\hat{\Phi}(x) - \lambda^+\, e^{-\alpha\,\mu} \int_{-\frac{\mu}{\sigma}}^{\infty} e^{-\alpha\sigma x}\, d\hat{\Phi}(x) \\
&= (\lambda^+ - \lambda^-)\,\hat{\Phi}\left(\frac{\mu}{\sigma}\right) - \lambda^- + \\
&\quad + e^{\frac{1}{2}\alpha^2\sigma^2}\left[\lambda^-\, e^{\alpha\,\mu}\,\hat{\Phi}\left(-\frac{\mu}{\sigma} - \alpha\sigma\right) - \lambda^+\, e^{-\alpha\,\mu}\,\hat{\Phi}\left(\frac{\mu}{\sigma} - \alpha\sigma\right)\right].
\end{aligned}
$$

For the last equality, we use that $\int_z^\infty e^{-\alpha\sigma x}\, d\hat{\Phi}(x) = e^{\frac{1}{2}\alpha^2\sigma^2}\,\hat{\Phi}\,(-\alpha\sigma - z)$.

(i)
From (0) we obtain

$$\partial_\mu V(\mu,\sigma) \;=\; \alpha\, e^{\frac{1}{2}\alpha^2\sigma^2}\left[\lambda^-\, e^{\alpha\mu}\,\hat{\Phi}\left(-\frac{\mu}{\sigma} - \alpha\sigma\right) + \lambda^+\, e^{-\alpha\mu}\,\hat{\Phi}\left(\frac{\mu}{\sigma} - \alpha\sigma\right)\right],$$

thus $\partial_\mu V(\mu,\sigma) > 0$ for all μ and σ.

(ii)
From (0) we obtain

$$\begin{aligned}\partial_\sigma V(\mu,\sigma) \;=\;& \alpha^2\,\sigma\; e^{\frac{1}{2}\alpha^2\sigma^2}\left[\lambda^-\, e^{\alpha\mu}\,\hat{\Phi}\left(-\frac{\mu}{\sigma} - \alpha\sigma\right) - \lambda^+\, e^{-\alpha\mu}\,\hat{\Phi}\left(\frac{\mu}{\sigma} - \alpha\sigma\right)\right]\\ &-\alpha\,(\lambda^- - \lambda^+)\,\hat{\varphi}\left(\frac{\mu}{\sigma}\right).\end{aligned}$$

It follows:

- $\partial_\sigma V(\mu,0) = 0$, using that $\hat{\Phi}(-\infty) = 0$, $\hat{\Phi}(\infty) = 1$ and $\hat{\varphi}(\infty) = 0$.
- Let us consider $f(\mu,\sigma) = \sigma^{-1}\, e^{-\frac{1}{2}\alpha^2\sigma^2}\, e^{-\alpha\,\mu}\,\partial_\sigma V(\mu,\sigma)$ for $\sigma > 0$. We show that $f(\mu,\sigma) < 0$.
  Suppose that for some $\mu^\star$ and $\sigma(\mu^\star) > 0$, $f(\mu,\sigma(\mu^\star)) > 0$. Since $f(\mu,\cdot)$ is continuous, $\lim_{\sigma\to 0} f(\mu,\sigma) = -\lambda^+\, e^{-2\,\alpha\,\mu} < 0$ and $\lim_{\sigma\to\infty} f(\mu,\sigma) = 0$ for all $\mu > 0$, we can assume without loss of generality that $\sigma(\mu^\star) > 0$ is a local maxima of $f(\mu^\star,\cdot)$. We compute the partial derivative of f with respect to σ. We have

  $$\begin{aligned}\partial_\sigma f(\mu,\sigma) \;=\;& \hat{\varphi}\left(\frac{\mu}{\sigma} + \alpha\sigma\right)\,\Big[\lambda^-(\mu\sigma^{-2} - \alpha) + \lambda^+(\mu\sigma^{-2} + \alpha)\\ &-\alpha^{-1}\,(\lambda^- - \lambda^+)\,\left(\mu^2\sigma^{-4} - \alpha^2 - \sigma^{-2}\right)\Big]\end{aligned}$$

  Let $\eta = \sigma^{-2}$, then

  $$\begin{aligned}\partial_\sigma f(\mu,\sigma) = 0 \;\;&\Leftrightarrow\;\; \eta\,\left[-\frac{\lambda^- - \lambda^+}{\alpha}\,\mu^2\,\eta + (\lambda^- + \lambda^+)\,\mu + \frac{(\lambda^- - \lambda^+)}{\alpha}\right] = 0\\ &\Leftrightarrow\;\; \eta \in \{0,\eta^\star(\mu)\}\end{aligned}$$

  where $\eta^\star(\mu) = \frac{\alpha\,\mu\,(\lambda^- + \lambda^+) + (\lambda^- - \lambda^+)}{(\lambda^- - \lambda^+)\,\mu^2}$. Moreover, for $\eta > \eta^\star(\mu)$, $\partial_\sigma f(\mu,\sigma) < 0$ and for $0 < \eta < \eta^\star(\mu)$, $\partial_\sigma f(\mu,\sigma) > 0$. It follows that $\sigma^\star(\mu) = \eta^\star(\mu)^{-1/2} > 0$ is the unique (local) maximum/minimun of $f(\mu,\cdot)$ and since for $\sigma > \sigma^\star(\mu)$, $\partial_\sigma f(\mu,\sigma) > 0$ and for $0 < \sigma < \sigma^\star(\mu)$, $\partial_\sigma f(\mu,\sigma) < 0$, $\sigma^\star(\mu)$ is a minimum. This contradicts the existence of $\mu^\star$ and $\sigma(\mu^\star)$ local maxima of $f(\mu^\star,\cdot)$ such that $f(\mu^\star,\sigma(\mu^\star)) > 0$. Hence, $f(\mu,\sigma) < 0$ and therefore $\partial_\sigma V(\mu,\sigma) < 0$.

- $\lim_{\sigma\to\infty} \partial_\sigma V(\mu,\sigma) = 0$ for $\mu > 0$ since $\lim_{\sigma\to\infty}\left(\sigma\, e^{\frac{1}{2}\sigma^2\alpha^2}\, e^{\alpha\mu}\, \hat{\Phi}(-\frac{\mu}{\sigma} - \alpha\sigma)\right) = \lim_{\sigma\to\infty}\left(\sigma\, e^{\frac{1}{2}\sigma^2\alpha^2}\, e^{-\alpha\mu}\, \hat{\Phi}(\frac{\mu}{\sigma} - \alpha\sigma)\right) = \frac{1}{\alpha\sqrt{2\pi}}$ and $\lim_{\sigma\to\infty} \hat{\varphi}(\frac{\mu}{\sigma}) = \frac{1}{\sqrt{2\pi}}$.

(iii)
Follows directly from (i) and (ii).

(iv)

$$
\begin{aligned}
S(\mu,\sigma) &= -\frac{\partial_\sigma V(\mu,\sigma)}{\partial_\mu V(\mu,\sigma)} \\
&= -\frac{\alpha^2\,\sigma\; e^{\frac{1}{2}\alpha^2\sigma^2}\left[\lambda^-\, e^{\alpha\mu}\,\hat{\Phi}\left(-\frac{\mu}{\sigma}-\alpha\sigma\right) - \lambda^+\, e^{-\alpha\mu}\,\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)\right]}{\alpha\, e^{\frac{1}{2}\alpha^2\sigma^2}\left[\lambda^-\, e^{\alpha\mu}\,\hat{\Phi}\left(-\frac{\mu}{\sigma}-\alpha\sigma\right) + \lambda^+\, e^{-\alpha\mu}\,\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)\right]} \\
&\quad + \frac{\alpha\,(\lambda^- - \lambda^+)\,\hat{\varphi}\left(\frac{\mu}{\sigma}\right)}{\alpha\, e^{\frac{1}{2}\alpha^2\sigma^2}\left[\lambda^-\, e^{\alpha\mu}\,\hat{\Phi}\left(-\frac{\mu}{\sigma}-\alpha\sigma\right) + \lambda^+\, e^{-\alpha\mu}\,\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)\right]} \\
&= -\frac{\alpha\,\sigma\left[\lambda^-\frac{\hat{\Phi}\left(-\frac{\mu}{\sigma}-\alpha\sigma\right)}{\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)} - \lambda^+\, e^{-2\,\alpha\mu}\right]}{\lambda^-\frac{\hat{\Phi}\left(-\frac{\mu}{\sigma}-\alpha\sigma\right)}{\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)} + \lambda^+\, e^{-2\,\alpha\mu}} + \frac{(\lambda^- - \lambda^+)\,\frac{\hat{\varphi}\left(\frac{\mu}{\sigma}+\alpha\,\sigma\right)}{\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)}}{\lambda^-\frac{\hat{\Phi}\left(-\frac{\mu}{\sigma}-\alpha\sigma\right)}{\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)} + \lambda^+\, e^{-2\,\alpha\mu}}.
\end{aligned}
$$

For $\frac{\mu}{\sigma}$ fix, we have

$$
\begin{aligned}
\lim_{\sigma\to\infty}\frac{\hat{\Phi}\left(-\frac{\mu}{\sigma}-\alpha\sigma\right)}{\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)} &= e^{-2\,\alpha\mu}, \\
\lim_{\sigma\to\infty}\frac{\hat{\varphi}\left(\frac{\mu}{\sigma}+\alpha\,\sigma\right)}{\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)} &= e^{-2\,\alpha\mu}\lim_{\sigma\to\infty}\frac{(\mu\,\sigma+\alpha\,\sigma^3)(-\mu+\alpha\,\sigma^2)}{\mu+\alpha\,\sigma^2}.
\end{aligned}
$$

and thus

$$
\lim_{\sigma\to\infty} S(\mu,\sigma) = \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+}\lim_{\sigma\to\infty}\left(-\alpha\,\sigma + \frac{(\mu\,\sigma+\alpha\,\sigma^3)(-\mu+\alpha\,\sigma^2)}{\mu+\alpha\,\sigma^2}\right) = \infty.
$$

(v)
Let consider the equation for S given above. For σ fix, we have

$$
\begin{aligned}
\lim_{\frac{\mu}{\sigma}\to\infty}\frac{\hat{\Phi}\left(-\frac{\mu}{\sigma}-\alpha\sigma\right)}{\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)} &= 0, \\
\lim_{\frac{\mu}{\sigma}\to\infty}\frac{\hat{\varphi}\left(\frac{\mu}{\sigma}+\alpha\,\sigma\right)}{\hat{\Phi}\left(\frac{\mu}{\sigma}-\alpha\sigma\right)} &= 0.
\end{aligned}
$$

and thus

$$\lim_{\frac{\mu}{\sigma}\to\infty} S(\mu,\sigma) \;\; = \;\; \alpha\,\sigma.$$

$\square$

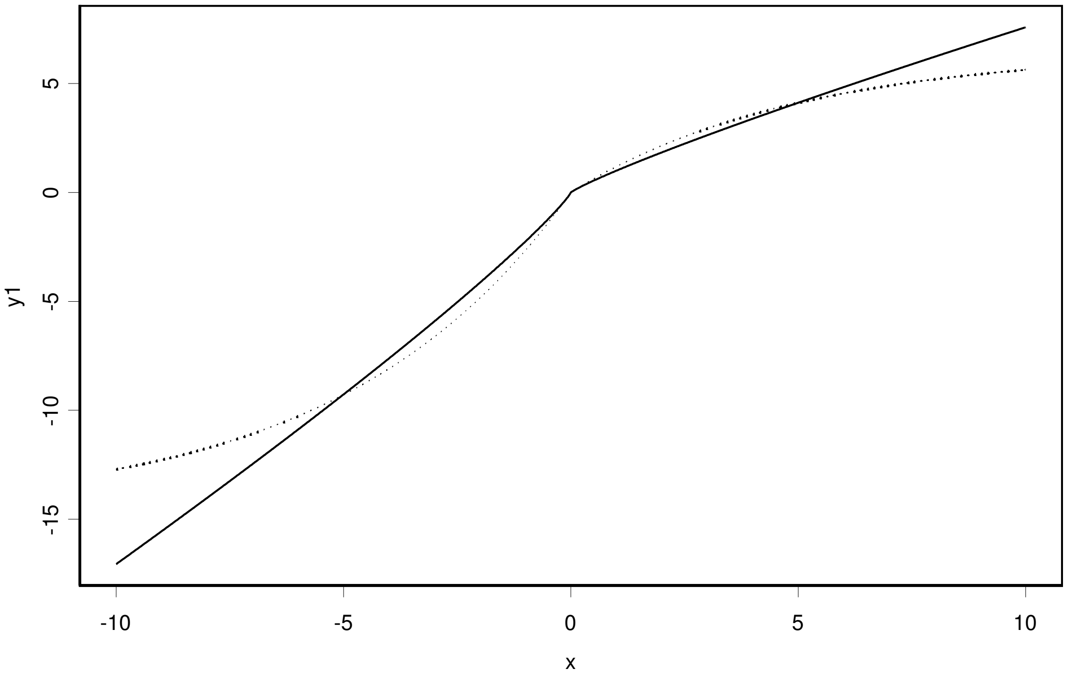

Figure 1: Tversky and Kahneman (1992) utility index (full line) and $u(x) = -\lambda^+ e^{-\alpha x} + \lambda^+$ for $x \geq 0$ and $u(x) = \lambda^- e^{\alpha x} - \lambda^-$ for $x < 0$ (dotted line), where $\lambda^+ = 6.52$, $\lambda^- = 14.7$ and $\alpha \approx 0.2$.

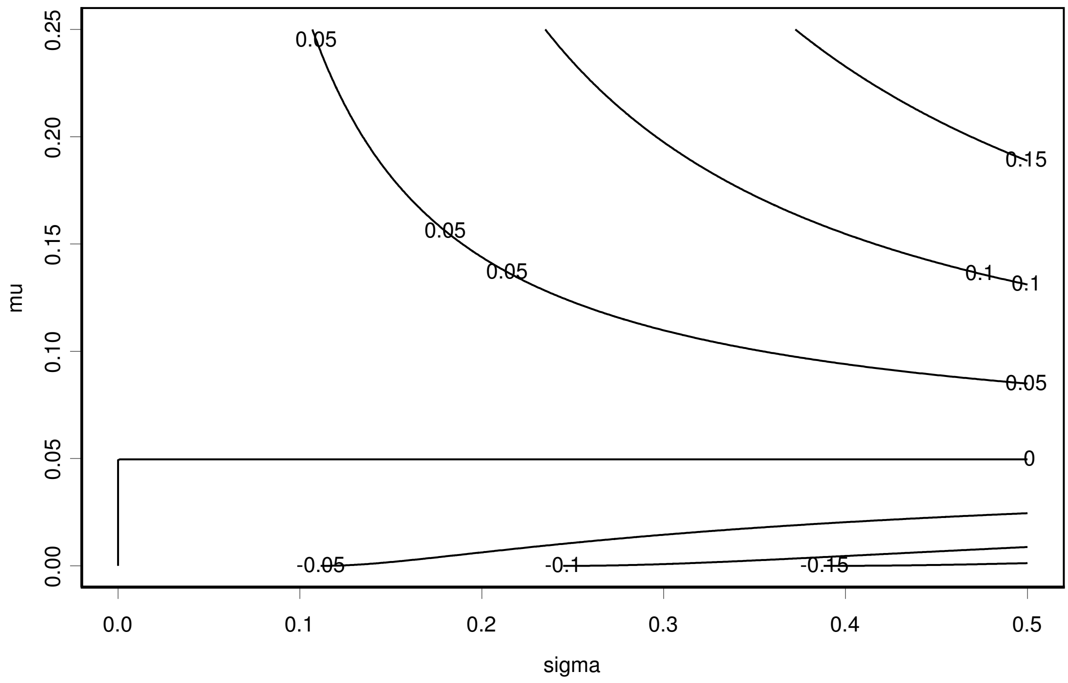

Figure 2: Indifference curves in the mean and standard deviation space for the utility function induced by Tversky and Kahnemann (1992) utility index and probability transformation.

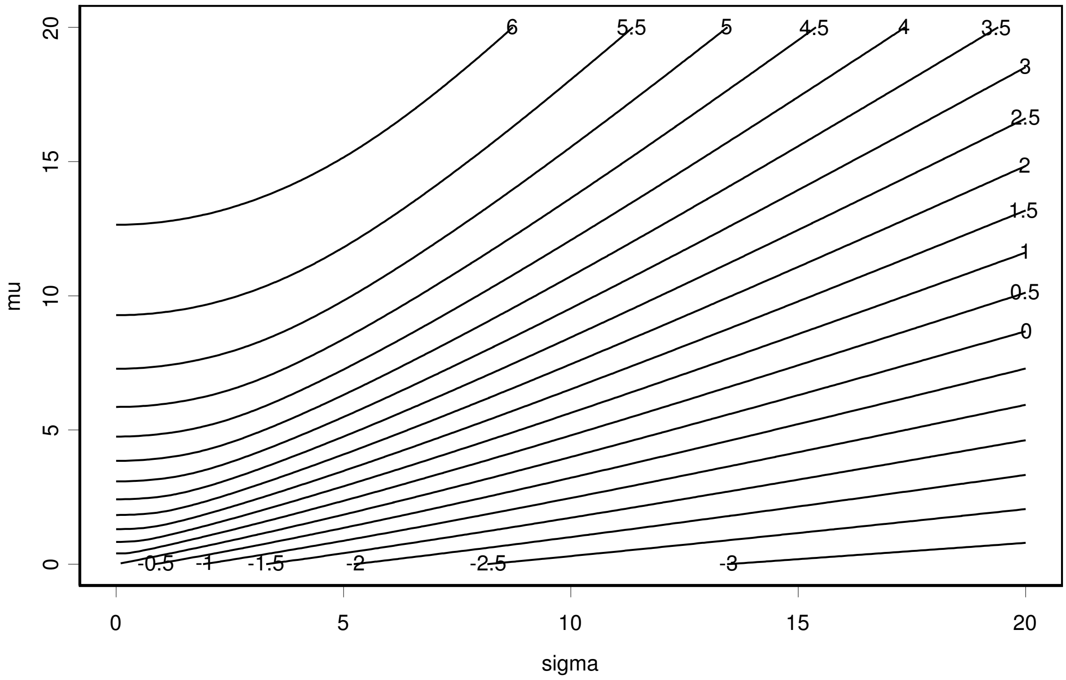

Figure 3: Indifference curves in the mean and standard deviation space for the utility function induced by $u(x) = -\lambda^+ e^{-\alpha x} + \lambda^+$ for $x \geq 0$ and $u(x) = \lambda^- e^{\alpha x} - \lambda^-$ for $x < 0$ where $\lambda^+ = 6.52$, $\lambda^- = 14.7$ and $\alpha \approx 0.2$.