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A Game-Theoretic Implication of the Riemann Hypothesis

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Abstract. The Riemann Hypothesis (RH) is one of the major unsolved problems in pure mathematics. In this note, a parameterized family of non-cooperative games is constructed with the property that, if RH holds true, then any game in this family admits a unique Nash equilibrium. We argue that this result is not degenerate. Indeed, neither is the conclusion a tautology, nor is RH used to define the family of games.

Keywords. Riemann hypothesis · Nash equilibrium

MSC-codes. 11, 91

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1. Introduction

The Riemann Hypothesis (RH) is a famous open problem in the field of analytic number theory. The purpose of this note is to report on a somewhat curious observation relating RH to the theory of games. Specifically, it is shown that a parameterized family of games may be constructed with the property that, provided that RH holds true, each game in this family admits a unique Nash equilibrium. The family of games constructed below falls roughly in the class of difference-form contests (Hirshleifer, 1989; Baik, 1998; Che and Gale, 2000; Ewerhart and Sun, 2018; Ewerhart, 2021). We argue that the example is not degenerate in a trivial way. I.e., neither is the conclusion of our result a tautology, nor is RH used in the definition of our class of games.

There does not seem to exist prior academic work that connects RH to the theory of games. Nobel Laureate John Nash, whose contributions in the early 50s became the basis of modern game theory (Nash, 1950, 1951) and who had also solved Hilbert's 19th problem on partial differential equations, is understood to have worked on RH.[1] However, the bibliography of Milnor (1998) does not list any manuscript written by Nash with an obvious relationship to number theory.[2]

The mathematical literature has come up with a large variety of conditions that are either necessary for, sufficient for, or equivalent to RH. In particular, Gröchenig (2020) related RH to the total positivity of a particular Fourier transform, and the observation made below draws heavily from his contribution.[3] However, the

[1] According to a popular biography (Nasar, 1998), as well as to a Hollywood movie based upon it, Nash's presentation on the topic at Columbia University in 1959 became incomprehensible because of his beginning mental illness (see also Sabbagh, 2003).

[2] In a volume coedited by late John Nash, Connes (2016) related RH to chip-firing games on graphs. Those games, however, are one-player "solitaire" problems (Baker and Norine, 2007). More recently, Carmona et al. (2020) recovered the GUE distribution from limits of equilibria in N-player stochastic games as $N \to \infty$. While GUE describes the distribution of distances between neighboring zeros of the Riemann zeta function (Montgomery, 1973; Odlyzko, 1987; Rudnick and Sarnak, 1994, 1996), GUE arises in numerous other applications as well.

[3] For an introduction to total positivity, see Karlin (1968). Katkova (2007) related RH to totally positive sequences.

present analysis also crucially exploits certain game-theoretic arguments that are not commonly discussed in the literature on the zeta function.[4]

The remainder of this paper is structured as follows. Section 2 provides background on RH. Section 3 introduces difference-form contests. Section 4 presents the main result. Section 5 offers some discussion. Section 6 outlines the proof of Theorem 1. Section 7 concludes. All proofs have been relegated to an Appendix.

2. Background on the Riemann hypothesis (RH)[5]

The *Riemann zeta function* is defined as

$$\zeta(s) = \sum\nolimits_{n=1}^{\infty} \frac{1}{n^s} \qquad (s > 1), \tag{1}$$

where n runs over all natural numbers and where $s > 1$ is required to guarantee convergence of the infinite sum (Titchmarch, 1986). With $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$ denoting the gamma function, one writes

$$\xi(s) = \tfrac{s(s-1)}{2} \pi^{-s/2} \Gamma(\tfrac{s}{2}) \zeta(s). \tag{2}$$

This function can be shown to admit an analytic representation that is valid for any argument s (even complex). Moreover, ξ satisfies the functional equation

$$\xi(s) = \xi(1 - s), \tag{3}$$

i.e., ξ is symmetric with respect to a reflection at $s = \frac{1}{2}$.

The ξ-function does not vanish for real values s. However, ξ is known to possess *complex* zeros of the form $s = \frac{1}{2} + i \cdot \tau$, where $i = \sqrt{-1}$ and $\tau \in \mathbb{R}$.[6] The *Riemann hypothesis (RH)*, formulated by Riemann (1859), claims that any zero

[4]These arguments concern conditions sufficient for the uniqueness of mixed strategy Nash equilibria in games with analytic payoffs (Ewerhart and Sun 2018; Ewerhart, 2021). Games with analytic payoff functions appear in early work on two-person zero-sum games on the square (Karlin, 1957, 1959). See also Ewerhart (2015) and Levine and Mattozzi (2022).

[5]This section may be skipped by readers interested only in the main results.

[6]E.g., $\rho_1 = 0.5 + i \cdot 14.134725\ldots$ is a zero of the ξ function, i.e., $\xi(\rho_1) = 0$.

of ξ is of this form. Expressed in terms of the zeta function, RH is commonly expressed by saying that "every non-trivial zero of ζ lies on the critical line." If true, the conjecture would admit powerful conclusions about the distribution of prime numbers (Davenport, 1980).

Proving RH is one of the seven problems for which the Clay Mathematics Institute awards a prize of one million dollars (Bombieri, 2000). Numerous interesting but ultimately partial results are available. For example, it is known that "more than 40 percent" of the non-tivial zeros of ζ lie on the critical line (Conrey, 1989). Moreover, starting with Turing (1953), substantial effort has been invested into attempts to reject RH using computational means. However, at least the first 10^{13} non-trivial zeros lie exactly on the critical line (Gourdon, 2004; Platt and Trudgian, 2021).[7] Still, as argued by Sarnak (2004, pp. 6-7), this need not mean that RH is "likely true." Finally, annoucements of alleged solutions to the problem are quite common (see, e.g., Schembri, 2018). At the time of writing, however, RH remains an open mathematical problem.

3. Difference-form contests

Two players, 1 and 2, each choose a nonnegative investment, $x_1 \geq 0$ and $x_2 \geq 0$. There is a prize of value $W > 0$. Expected payoffs are given by

$$\Pi_1(x_1, x_2) = F(x_1 - x_2)W - x_1, \quad (4)$$

$$\Pi_2(x_1, x_2) = F(x_2 - x_1)W - x_2, \quad (5)$$

where $F(t) = \frac{1}{2} + \int_0^t f(\tau)d\tau$ for some measurable function f that is symmetric with respect to the origin. The resulting two-player game will be denoted by $G_0 \equiv G_0(f, W)$.

It should be noted that the above definition does not require f to be a probability density function. If f is a probability density function, however, then $F(t) =$

[7] For details on the methods employed to ensure that all complex zeros of ζ up to a given height lie *exactly* on the critical line, see Edwards (1974, Ch. 8).

$\int_{-\infty}^{t} f(\tau)d\tau$ is the corresponding distribution function. Moreover, $F(x_1 - x_2)$ is the probability that player 1 wins the prize, while $F(x_2 - x_1) = 1 - F(x_1 - x_1)$ is the probability that player 2 wins the prize. In that case, therefore, G_0 is a symmetric *difference-form contest.* If f is not a probability density, then G_0 is not a difference-form contest in the usual meaning of the term.

Further, it should be noted that the above payoff functions need, in general, not be quasiconcave, even if f is a very well-behaved probability density.[8] It is therefore natural to allow for mixed strategies, defined as probability distributions on the Borel subsets of a suitably chosen compact subinterval of the real line (Dasgupta and Maskin, 1986). Pure strategies may then be considered as degenerate probability distributions, as usual.

The following examples illustrate equilibria in difference-form contests.

Example 1 (Hirshleifer, 1989; Baik, 1998). Let f be a continuous density single-peaked at the origin. If $W \leq 1/f(0)$, then players' expected payoffs are strictly declining in their own strategy. In that case, therefore, the unique Nash equilibrium of G_0 is in pure strategies, with equilibrium strategies given by $x_1^* = x_2^* = 0$.

Example 2 (Ewerhart and Sun, 2018). Let f be given as the logistic density $f(t) = \frac{\alpha \exp(-\alpha t)}{(1+\exp(-\alpha t))^2}$, where $\alpha = 6.75$. Suppose also that $W = 1$. Then, the unique mixed-strategy equilibrium in G_0 is symmetric, and has each player i independently choose $x_i = y_1 \equiv 0.45597$ with probability $q_1 = 0.51011$, and $x_i = 0$ with probability $q_2 = 1 - q_1$. The equilibrium payoff is $\Pi_i^* = 0.2674$. Player i's expected payoff against the equilibrium strategy,

$$E[\Pi_i(x_1, x_2)] = \frac{q_1}{1 + \exp(-\alpha(x_i - y_1))} + \frac{q_2}{1 + \exp(-\alpha x_i)} - x_i, \tag{6}$$

considered as a function of x_i, is depicted in Figure 1. As can be seen, its maxima are located at $x_i = y_1$ and $x_i = 0$.

[8] For illustration, see Example 2 below.

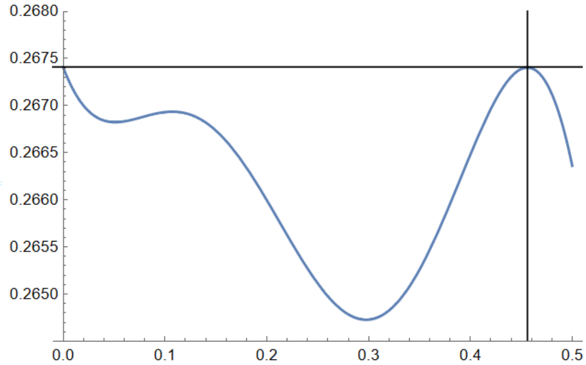

Figure 1. Expected payoff against the equilibrium strategy.

In general, however, the equilibrium of a difference-form contest need not be unique. We illustrate the possibility of multiple equilibria by modiyfing the assumptions in the framework of Che and Gale (2000).

Example 3. Let f be the uniform density over the interval $[-c, c]$, for some $c > 0$. Thus, $f(t) = \frac{1}{2c}$ if $t \in [-c, c]$, and $f(t) = 0$ otherwise. Then, $x_1^* = c$ and $x_2^* = 0$ form an asymmetric Nash equilibrium in pure strategies in G_0 in the *non-generic* case $W = 2c$.[9] By symmetry, a second equilibrium is given by $(x_1^{**}, x_2^{**}) = (0, c)$.

A way to understand the multiplicity in Example 3 is that the uniform density is not sufficiently well-behaved. Indeed, it is neither analytic nor a proper Pólya frequency function.

[9] Indeed, with

$$F(t) = \begin{cases} 0 & \text{if } t < -c \\ \frac{1}{2} + \frac{t}{2c} & \text{if } t \in [-c, c] \\ 1 & \text{if } t > c, \end{cases} \tag{7}$$

expected payoffs against the respective opponent's equilibrium strategy are given by

$$\Pi_1(x_1, 0) = 2cF(x_1) - x_1 \tag{8}$$

$$= \begin{cases} c & \text{if } x_1 \in [0, c] \\ 2c - x_1 & \text{if } x_1 > c, \end{cases} \tag{9}$$

$$\Pi_2(c, x_2) = 2cF(x_2 - c) - x_2 \tag{10}$$

$$= \begin{cases} 0 & \text{if } x_2 \in [0, 2c] \\ 2c - x_2 & \text{if } x_2 > 2c. \end{cases} \tag{11}$$

Therefore, $(x_1^*, x_2^*) = (c, 0)$ is indeed an equilibrium.

4. Statement of the main result

The statement of our main result uses the Riemann ξ-function, which has been introduced in Section 2. Gröchenig (2020) noted that the Fourier transform of the reciprocal of the "shifted" ξ-function exists for all frequencies. Building on his result, we can make the following observation.

Lemma 1 *The function*

$$f^{\#}(t) = \frac{\xi(\frac{1}{2})}{2\pi} \int_{-\infty}^{+\infty} \frac{\cos(tx)dx}{\xi(\frac{1}{2}+x)} \tag{12}$$

is well-defined for all $t \in \mathbb{R}$. *Moreover,* $f^{\#}$ *is symmetric with respect to the origin, continuous, and vanishing at infinity.*

Proof. See the Appendix. □

Clearly, $f^{\#}$ does not belong to the class of functions commonly employed in economics and statistical analysis (Johnson et al., 1995). In fact, little definite is known about $f^{\#}$. For instance, it is not even known if $f^{\#}$ is globally nonnegative. However, the numerically obtained graph of $f^{\#}$, which is outlined in Figure 2, suggests that $f^{\#}$ is, in fact, a very well-behaved probability density function.[10]

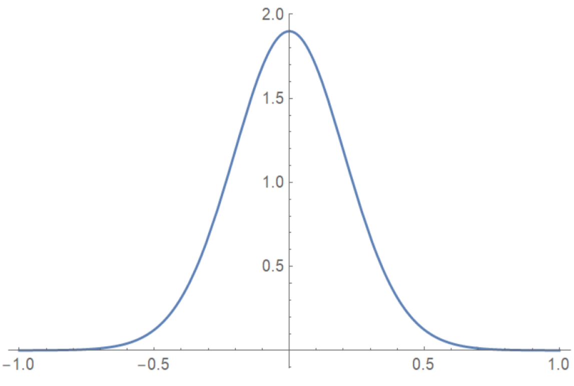

Figure 2. The graph of $f^{\#}$.

[10] All computations have been conducted using Wolfram's *Mathematica* 12.0.0 Kernel for Microsoft Windows (64-bit).

The generalized difference-form contest in which f is chosen to be $f^\#$ will be denoted by $G_0^\# \equiv G_0(f^\#, W)$. We are ready to state our main result.

Theorem 1. *If RH holds true, then $G_0^\#$ admits precisely one equilibrium (for any $W > 0$).*

Proof. See Section 6. □

5. Discussion

In this section, we will argue that Theorem 1 is not degenerate in an obvious way, because the conclusion is not a tautology (Subsection 5.1) and because RH does not enter the definition of the game (Subsection 5.2).

5.1 The conclusion of Theorem 1 is not a tautology

It should be noted that it is not at all difficult to come up with a non-cooperative game that has precisely one Nash equilibrium if RH holds true. For example, the Prisoner's Dilemma admits a unique Nash equilibrium if RH holds true. As the conclusion is true, the implication holds regardless of the validity of the premise (by the rules of Boolean logic). But the situation is different here. In view of the possibility of multiple equilibria illustrated by Example 3, it is not known (and might never become known) if the conclusion of Theorem 1 holds true or not. Thus, the conclusion is an open conjecture like the hypothesis.[11] In particular, if the conclusion of equilibrium uniqueness could be shown to be wrong (which is not feasible in the case of the Prisoner's Dilemma), then the hypothesis would be proven wrong.[12]

[11] Both RH and equilibrium uniqueness in the two-player contest may be characterized as being undecidable in the *practical* sense. Undecidability in the *logical* sense is a possibility here as well (i.e., RH and/or equilibrium uniqueness might be true but not provable), but this possibility is not crucial for the present discussion.

[12] A similar type of reasoning is used in the literature on the P versus NP problem in computational complexity theory (Cook, 1971), which happens to be a millennium problem like RH.

5.2 RH does not enter the definition of the game

It would be nice, and certainly more satisfying, to find a game-theoretic conjecture that is logically equivalent to RH. In the abstract, this is actually not a big problem. E.g., one may even easily write down games for which the existence of a unique Nash equilibrium is equivalent to RH. To see this, consider the game G_1 depicted in Figure 3, where

$$\theta = \begin{cases} +1 & \text{if RH holds true} \\ -1 & \text{if RH does not hold true.} \end{cases} \tag{13}$$

If RH holds true, then G_1 admits (T,L) as the unique Nash equilibrium. If, however, RH does not hold true, then there are two Nash equilibria in pure strategies, viz (T,R) and (B,L). Similarly, G_2 admits (T,L) as a unique pure-strategy Nash equilibrium if RH holds, and otherwise no pure-strategy Nash equilibrium.

G_1	L	R
T	θ,θ	0,0
B	0,0	-1,-1

G_2	L	R
T	$1,\theta$	$\theta,0$
B	0,1	0,0

Figure 3. The games G_1 and G_2.

In such examples, however, RH is used directly in the description of the payoff functions. That is, even if the strategy chosen by player 2 is correctly anticipated in G_1 or G_2, a human player 1 could not tell if T yields a higher payoff than B. In contrast, RH has no role in the definition of payoffs in $G_0^{\#}$, i.e., expected payoffs could, at least in principle, be approximated up to arbitrary accuracy without assuming RH. In fact, given that best responses in the mixed extension of $G_0^{\#}$ have finite support, this argument extends to the relevant class of randomized strategies. For this reason, the two-player contest might be a more appealing example than G_1 or G_2, even though Theorem 1 does not capture a logical equivalence.

6. Proof of Theorem 1

The proof has three steps. First, it is shown that, provided that RH holds true, the function $f^{\#}$ defined through Lemma 1 is both a proper Pólya frequency function and a probability density function. In a second step, $f^{\#}$ is seen to be analytic on the real line (irrespective of RH). Finally, it is shown that, because $f^{\#}$ satisfies these properties, $G_0^{\#}$ admits a unique Nash equilibrium if RH holds true.

Lemma 2. *Suppose that RH holds true. Then, $f^{\#}$ is a both a proper Pólya frequency function and a probability density function.*

Proof. See the Appendix. □

The property of f being a *proper Pólya frequency function* means that, in addition to f being integrable over $\mathbb{R}$, it is the case that, for any $n \geq 1$, and for any real parameters $a_1 > a_2 > \cdots > a_n$ and $b_1 > b_2 > \cdots > b_n$, the matrix

$$M_f \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} = \begin{pmatrix} f(a_1 - b_1) & f(a_1 - b_2) & \cdots & f(a_1 - b_n) \\ f(a_2 - b_1) & f(a_2 - b_2) & \cdots & f(a_2 - b_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(a_n - b_1) & f(a_n - b_2) & \cdots & f(a_n - b_n) \end{pmatrix} \tag{14}$$

has a positive determinant. The requirements for $n = 1$ and $n = 2$ can be shown to correspond to positivity and strict logconcavity of f, respectively, but the requirements for $n \geq 3$ do not admit an equally simple interpretation. Examples of proper Pólya frequency functions include any logistic or normal probability density function (Karlin, 1957, 1959).

Lemma 2 is based upon a mathematically deep correspondence between *Pólya frequency functions* (where the determinant in the above definition is merely required to be nonnegative, and there must be at least two points where the function does not vanish) and functions of the *Laguerre-Pólya class of type II (* i.e., functions analytic on the complex plane that are locally uniform limits, not identically zero, of polynomials whose zeros all lie on the real line). This correspondence was established by Schoenberg (1947) who observed that Pólya frequency functions may be characterized as normally smoothed limits of convolutions of multiple exponential probability density functions. Since the Laplace transform of an exponential probability density function is just the reciprocal of a factor in a Weierstrass product representation (Schoenberg, 1951, pp. 349-350), and since the Laplace transform converts convolutions into products, the correspondence connects Pólya frequency functions with functions of the Laguerre-Pólya class. In particular, as noted by Gröchenig (2020, Thm. 4), the shape of $f^{\#}$ allows conclusions regarding the location of the zeros of the ξ-function, and vice versa. More specifically, $f^{\#}$ is a Pólya frequency function if and only if RH holds true. Lemma 2 refines the sufficiency part of that observation using a result by Schoenberg and Whitney (1953) for *proper* Pólya frequency functions.

Next, we show that $f^{\#}$ is real-analytic, i.e., the derivatives of $f^{\#}$ of all finite orders exist, and at any point on the real line, $f^{\#}$ is locally approximated arbitrarily well by its Taylor expansion.

Lemma 3. *$f^{\#}$ is analytic on $\mathbb{R}$.*

Proof. See the Appendix. □

The proof of Lemma 3 exploits, like Lemma 1, that the reciprocal of the shifted ξ function diminishes exponentially for real arguments large in absolute value.

The third step of the proof exploits the following result, whose uniqueness part is a special case of Ewerhart (2021, Prop. 1).

Lemma 4. *Suppose that f is a probability density function that is symmetric with respect to the origin. Then, a symmetric equilibrium μ^* exists in G_0. If, in addition, f is a proper Pólya frequency function as well as analytic on $\mathbb{R}$, then there are no asymmetric equilibria, and μ^* is the unique equilibrium.*

Proof. See the Appendix.

The existence part of the proof of Lemma 4 is based on standard conditions for symmetric compact games with continuous payoffs (Becker and Damianov, 2006). In contrast, the uniqueness part has multiple steps, in which the various assumptions are combined with powerful theorems from complex analysis and the theory of two-person zero-sum games. In view of Lemma 2, however, it seems most desirable to understand *how* the property of f being a proper Pólya frequency function contributes to the conclusion that the equilibrium is unique. To explain this, we recall that general properties of contests with analytic payoffs imply the existence of a finite set S of pure strategies with the property that *any* mixed equilibrium strategy is bound to randomize over this set.

Then, given the finite set of potential maximizers, necessary first-order conditions hold at all positive investment levels used with positive probability in equilibrium. Moreover, if a player chooses a zero investment level with positive probability (this is always the case for both players, as a consideration of second-order conditions shows), then the resulting equilibrium payoff Π^* must necessarily be the same as that resulting from the lowest positive investment level used with positive probability (and if there is no such positive investment level, then there is nothing to show). These first-order conditions and the indifference condition may be combined into a system of linear equations, one for each player, in which the probabilities with which pure strategies are used and the player's equilibrium payoff are the unknowns. To prove uniqueness, it then suffices to show that this system of linear equations is not degenerate.

To this end, the propery of f being a proper Pólya frequency function is exploited. In fact, given that f features prominently in the first-order conditions, this last point would be straightforward if it were known that all investment levels used in equilibrium with positive probability are positive. In that case, the unique solvability of the linear system of first-order conditions would follow directly from the fact that the relevant determinant is nonzero (even positive). However, as mentioned above, both players necessarily choose the zero investment level with positive probability, which is why this simple argument does not go through. Instead, the linear system of equations contains an indifference condition, which involves the distribution function F in addition to the density function f. But then, fortunately, the sign of the relevant determinant may still be evaluated as an integral over positive determinants.

After these preparations, the proof of Theorem 1 is straightforward. Suppose that RH holds true. Then, by Lemmas 2 and 3, $f^{\#}$ satisfies the assumptions of Lemma 4. Hence, there exists a unique equilibrium in $G^{\#}$.

7. Concluding remarks

We have constructed a parameterized family of two-person games with the property that every game in this family admits a unique Nash equilibrium if RH holds true. Thus, a game-theorist able to identify two equilibria in one of the considered games would reject RH. We have done our best to explain why we believe that our result is not degenerate in an obvious way.[13]

It would be desirable to better understand how tight the conditions of Theorem 1 are for equilibrium uniqueness. For example, as suggested by an anonymous referee, it would be interesting to know if the existence of a non-trivial zero of the Riemann zeta function slightly off the critical line, possibly with a very large absolute imaginary component, would allow to construct multiple equilibria in some game $G_0^{\#}$. This question, however, must be left for future work.

[13] On a more speculative note, the steps of the analysis could be replicated for more general classes of L-functions (Sarnak, 2004). However, additional assumptions might be needed, such as that the L-function does not vanish at $s = \frac{1}{2}$ (Stark and Zagier, 1980).

A. Appendix

We start with some auxiliary results. The following lemma is well-known.

Lemma A.1 *The Riemann ξ-function, defined in (2), has the following properties:*

(i) $\xi > 0$ *on the real line, in particular* $\xi(\frac{1}{2}) > 0$;

(ii) $\ln \xi(s) \sim \frac{1}{2} s \ln s$ *for* s *real and* $s \to \infty$;

(iii) $\operatorname{Re} \rho \in [0, 1]$ *for any zero* ρ *of* ξ;[14]

(iv) $\sum_\rho \frac{1}{|\rho|^2} < \infty$, *where the sum runs over all zeros of* ξ;

(v) $\sum_\rho \frac{1}{|\rho|}$ *diverges.*

Proof. For claims (i) through (iii), see Titchmarch (1986, pp. 29-30). For claims (iv) and (v), see Davenport (1980, Sec. 12). □

The following result, for which we could not find a reference, is used in the proof of Lemma 1.

Lemma A.2 *The "shifted" ξ-function admits the product representation*

$$\xi(s + \tfrac{1}{2}) = \xi(\tfrac{1}{2}) \prod_\rho \left(1 - \frac{s}{\rho - \frac{1}{2}}\right) e^{s/(\rho - \frac{1}{2})}, \tag{15}$$

where the product runs over the non-trivial zeros of the zeta function. Moreover, $\sum_\rho \left|\rho - \frac{1}{2}\right|^{-2} < \infty$ *and* $\sum_\rho \left|\rho - \frac{1}{2}\right|^{-1} = \infty$.

Proof. We start with the first claim. By Edwards (1974, Sec. 2.8), the ξ-function admits the product representation

$$\xi(s) = c \cdot \prod_\rho \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right), \tag{16}$$

[14] As usual, $\operatorname{Re} \rho$ and $\operatorname{Im} \rho$ denote the real and imaginary parts of the complex number ρ, respectively.

where c is a constant, and it is understood that the factors for ρ and $1-\rho$ are paired to guarantee conditional convergence. Replacing s by $s+\frac{1}{2}$, one obtains

$$\xi(s+\tfrac{1}{2}) = c \cdot \prod_\rho \left(1 - \frac{s}{\rho - \frac{1}{2}}\right). \tag{17}$$

Making the pairing explicit yields

$$\xi(s+\tfrac{1}{2}) = c \cdot \prod_{\operatorname{Im}\rho>0} \left(1 - \frac{s}{\rho - \frac{1}{2}}\right)\left(1 - \frac{s}{(1-\rho) - \frac{1}{2}}\right), \tag{18}$$

where we used the fact that $\operatorname{Im}\rho$ and $\operatorname{Im}(1-\rho)$ have opposite signs. Noting that $e^{s/(\rho-\frac{1}{2})}e^{s/(1-\rho-\frac{1}{2})} = 1$, relationship (18) transforms into

$$\begin{aligned} \xi(s+\tfrac{1}{2}) &= c \cdot \prod_{\operatorname{Im}\rho>0} \left(1 - \frac{s}{\rho - \frac{1}{2}}\right) e^{s/(\rho-\frac{1}{2})} \left(1 - \frac{s}{1-\rho - \frac{1}{2}}\right) e^{s/(1-\rho-\frac{1}{2})} && (19) \\ &= c \cdot \prod_\rho \left(1 - \frac{s}{\rho - \frac{1}{2}}\right) e^{s/(\rho-\frac{1}{2})}, && (20) \end{aligned}$$

where the product in (20) is, at this point, still understood to pair the factors for ρ and $1-\rho$. However, by Edwards (1974, Sec. 2.5), $\sum_\rho \left|\rho - \frac{1}{2}\right|^{-2} < \infty$. Hence, using the Weierstrass Factorization Theorem for entire functions (Conway, 1978, p. 279), the product in (20) converges locally uniformly on the complex plane. Moreover, letting $s = 0$ in (20) yields $c = \xi(\frac{1}{2})$. This completes the proof of the first claim. The second claim has been shown above. As for the third and final claim, we note that by Lemma A.1(iii), $\operatorname{Re}\rho \in [0,1]$. Moreover, by Lemma A.1(iv), $|\rho|^2 > \frac{1}{4}$ for all but finitely many zeros ρ.[15] Hence, for all but finitely many zeros ρ,

$$\left|\rho - \tfrac{1}{2}\right|^2 = \left|(\operatorname{Re}\rho) - \tfrac{1}{2}\right|^2 + |\operatorname{Im}\rho|^2 \le \tfrac{1}{4} + |\operatorname{Im}\rho|^2 \le 2\,|\rho|^2, \tag{21}$$

which implies $\left|\rho - \frac{1}{2}\right| \le \sqrt{2}\,|\rho|$. Therefore, using Lemma A.1(v), the infinite sum $\sum_\rho \left|\rho - \frac{1}{2}\right|^{-1}$ diverges. This proves the lemma. □

Proof of Lemma 1. As noted by Gröchenig (2020, p. 4), the exponential decline of $1/\xi(x+\frac{1}{2})$ on the real line ensures that its Fourier transform exists, i.e., the

[15] Computationally, this inequality holds of course for all zeros ρ, because $\operatorname{Im}\rho_1 \simeq 14.134725$ for the zero ρ_1 with the smallest positive imaginary component.

integral

$$\widetilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(itx)dx}{\xi(\frac{1}{2}+x)} \qquad (-\infty < t < \infty) \tag{22}$$

is well-defined. Now, using Euler's formula

$$\exp(itx) = \cos(tx) + i\sin(tx), \tag{23}$$

as well as the functional equation (3), one observes that

$$\widetilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\cos(tx)}{\xi(\frac{1}{2}+x)} dx + \frac{i}{2\pi} \underbrace{\int_{-\infty}^{+\infty} \frac{\sin(tx)}{\xi(\frac{1}{2}+x)} dx}_{=0} \tag{24}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\cos(tx)}{\xi(\frac{1}{2}+x)} dx. \tag{25}$$

Thus, the integral in (12) is well-defined, and so is $f^{\#}$. Next, we note that $f^{\#}$, defined as a cosine integral transform, is obviously symmetric with respect to the origin. Finally, being the Fourier transform of an absolutely integrable function, $\widetilde{f}$ is continuous on $\mathbb{R}$ and vanishes at infinity (Rudin, 1974, Thm. 9.6). Both properties are inherited by $f^{\#} = \xi(\frac{1}{2})\widetilde{f}$, of course. This proves the lemma. □

Proof of Lemma 2. Suppose that RH holds true. Then, by Gröchenig (2020, Thm. 3), there exists a Pólya frequency function $\Lambda(x)$ such that

$$\frac{1}{\xi(\frac{1}{2}+is)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Lambda(x) \exp(xs) dt \qquad (-\infty < s < \infty). \tag{26}$$

Applying now Mellin's inverse formula, as in Schönberg (1947, Thm. 3), one obtains

$$\Lambda(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\exp(xz)dz}{\xi(\frac{1}{2}+iz)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(ixt)dt}{\xi(\frac{1}{2}+t)} = \widetilde{f}(x). \tag{27}$$

Clearly, this is just the sufficiency part of Gröchenig's (2020, Thm. 4), i.e., that $\widetilde{f}$ is a Pólya frequency function if RH holds true. We claim, however, that $\widetilde{f}$ is even a *proper* Pólya frequency function if RH holds true. We know that the shifted ξ is an entire function. Moreover, for each non-trivial root ρ of the zeta function,

$\rho - \frac{1}{2} = i \operatorname{Im} \rho$ is purely imaginary. Hence, invoking Lemma A.2, we have the product representation

$$\xi(\tfrac{1}{2} + is) = \xi(\tfrac{1}{2}) \prod_\rho \left(1 - \frac{s}{\operatorname{Im} \rho}\right) e^{s/\operatorname{Im} \rho}, \tag{28}$$

where $\sum_\rho |\operatorname{Im} \rho|^{-1} = \infty$ and $\sum_\rho |\operatorname{Im} \rho|^{-2} < \infty$. Therefore, in view of (26) and (27), Schoenberg and Whitney (1953, Thm. 1, Case 2), implies that $\widetilde{f}$ is a proper Pólya frequency function. Given that $\xi(\frac{1}{2}) > 0$ by Lemma A.1(i), the same holds true for $f^{\#}$. It remains to be shown that $f^{\#}$ is a probability density function. By evaluating the determinant of (14) in the special case $n = 1$, $a_1 = t$, and $b_1 = 0$, one checks that $f^{\#}$ is globally positive. Moreover, from the Fourier inversion theorem (Rudin, 1974, Thm. 9.11),

$$\int_{-\infty}^{+\infty} \widetilde{f}(t) dt = \frac{1}{\xi(\frac{1}{2})}. \tag{29}$$

Thus, $\int_{-\infty}^{+\infty} f^{\#}(t) dt = 1$, and $f^{\#}$ is indeed a probability density function. $\square$

Proof of Lemma 3. Take some $\varepsilon > 0$. For $z \in \mathbb{C}$ such that $|\operatorname{Im}(z)| < \varepsilon$, we have

$$\left| \frac{\exp(izx)}{\xi(\frac{1}{2} + x)} \right| \leq \frac{\exp(\varepsilon |x|)}{\xi(\frac{1}{2} + x)} \tag{30}$$

Moreover, from Lemma A.1(ii),

$$\frac{\exp(\varepsilon |x|)}{\xi(\frac{1}{2} + x)} = \mathcal{O}\left(\exp\left(|x| \left(\varepsilon - \frac{\ln x}{2}\right)\right)\right). \tag{31}$$

Focusing on the case $|x| \geq \exp(2\varepsilon)$, one observes that the left-hand side of (31) is asymptotically diminishing at an exponential rate as $|x| \to \infty$, i.e.,

$$\frac{\exp(\varepsilon |x|)}{\xi(\frac{1}{2} + x)} = \mathcal{O}\left(\exp\left(-\varepsilon |x|\right)\right). \tag{32}$$

The analytic nature of $\widetilde{f}$ on the strip $|\operatorname{Im}(z)| < \varepsilon$ may now be deduced directly from (32) using the conditions put forward by Mattner (2001).[16] This proves the lemma. $\square$

[16] Alternatively, one may rely on Paley and Wiener (1934, Thm. I).

Proof of Lemma 4. *(Existence)* By assumption, f is a probability density function. Therefore, any strategy $x_i > W$ is strictly dominated by $x_i = 0$. We may therefore, without loss of generality, assume that each player's strategy space is $[0, W]$. Given that payoff functions are continuous, it follows from Becker and Damianov (2006, Thm. 1) that G_0 admits a symmetric mixed-strategy Nash equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$, where $\mu_1^* = \mu_2^*$. *(Uniqueness)* We check the conditions of Ewerhart (2021, Prop. 1).[17] By assumption, f is a proper Pólya frequency function. Hence, f is a Pólya frequency function. Using Schoenberg (1951, Lemma 1), this implies that f is logconcave. But given analyticity, f is differentiable. Moreover, $f > 0$ because f is a proper Pólya frequency function. Therefore, f'/f is weakly declining. Next, one notes that $f'(0) = 0$ because f is symmetric with respect to the origin. But, since f is analytic on $\mathbb{R}$, so is f'. Further,f' is not constant (otherwise f would be affine, in conflict with the assumption that f is a probability density function). Hence, $t = 0$ is an isolated zero of f'. Combining this with the fact that f'/f is weakly declining implies that $f'(t)/f(t) < 0$ for all $t > 0$, and $f'(t)/f(t) > 0$ for all $t < 0$. This means that all conditions of Ewerhart (2021, Prop. 1) are satisfied. Thus, the equilibrium is indeed unique. □

For the reader's convenience, the material below has been replicated from Ewerhart (2021). Compared to the original contribution, however, the proof below is substantially shorter because the difference-form contest is known to be symmetric in the present analysis.

Self-contained proof of the uniqueness part of Lemma 4. Fix $W > 0$. As shown above, there is at least one (even symmetric) mixed-strategy Nash equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$ in G_0. Given that F is analytic on the real line, a straightforward extension of Ewerhart and Sun (2018, Lemma 1) shows that there is a finite set S of pure strategies such that any pure best response to μ_1^* is contained in

[17] A self-contained proof is added below.

S. In a nutshell, expected payoffs can be shown to be analytic but not constant, which implies the claim. Next, we note that μ_2^* is a mixed best response to μ_1^*. Hence, the support of μ_2^* is finite and contained in S. Suppose there exists another equilibrium $\mu^{**} = (\mu_1^{**}, \mu_2^{**})$ in $G_0^{\#}$. By subsidizing each player with the investment of the other player, the game G_0 is seen to be strategically equivalent to a two-person zero-sum game, which implies the exchangeability of Nash equilibrium strategies (Ewerhart, 2017, Lemma A.1).[18] Therefore, μ_2^* is a best response also to μ_1^{**}. Consider the set of pure strategies $S' = \{y_1 > y_2 > \ldots > y_K \geq 0\}$ in the support of either μ_1^* or μ_1^{**}.[19] If $K = 1$, then $\mu_1^* = \mu_1^{**}$ and we are done. Suppose, therefore, that $k \geq 2$. Equating marginal benefits with marginal costs at the certainly positive levels of investment used in equilibrium $y_1, \ldots, y_{K-1}$, we have the necessary first-order conditions,

$$\sum_{\widehat{k}=1}^{K} q_{\widehat{k}} f(y_k - y_{\widehat{k}}) W = 1 \qquad (k = 1, \ldots, K-1). \tag{33}$$

Moreover, we know that

$$\sum_{\widehat{k}=1}^{K} q_{\widehat{k}} F(y_k - y_{\widehat{k}}) W = \Pi^* \qquad (k \in \{K-1, K\}), \tag{34}$$

where Π^* is the equilibrium payoff resulting from μ^* (analogous conditions for $k \in \{1, \ldots, K-2\}$ are not needed, neither is the accounting equation $\sum_{\widehat{k}=1}^{K} q_{\widehat{k}} = 1$ needed). Combining the $(K-1)$ first-order conditions (33) with the two payoff conditions (34), we arrive at the system

$$M \begin{pmatrix} q_1 \\ \vdots \\ q_K \\ -\Pi^*/W \end{pmatrix} = \frac{1}{W} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ y_{K-1} \\ y_K \end{pmatrix} \in \mathbb{R}^{K+1}, \tag{35}$$

[18] In contrast to zero-sum games, however, this does not imply payoff equivalence across equilibria in G_0.

[19] As mentioned in the body of the paper, one can show that $y_K = 0$. However, that result is not needed in the proof.

with the square matrix $M_1 \in \mathbb{R}^{(K+1)\times(K+1)}$ defined by

$$M_1 = \begin{pmatrix} \underbrace{f(y_1 - y_1)}_{=f(0)} & \cdots & f(y_1 - y_{K-1}) & f(y_1 - y_K) & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ f(y_{K-1} - y_1) & \cdots & \underbrace{f(y_{K-1} - y_{K-1})}_{=f(0)} & f(y_{K-1} - y_K) & 0 \\ F(y_{K-1} - y_1) & \cdots & \underbrace{F(y_{K-1} - y_{K-1})}_{=1/2} & F(y_{K-1} - y_K) & 1 \\ F(y_K - y_1) & \cdots & F(y_K - y_{K-1}) & \underbrace{F(y_K - y_K)}_{=1/2} & 1 \end{pmatrix}. \tag{36}$$

We claim that this system admits at most one solution. Indeed, subtracting the last row from the second-to-last row leads to

$$\det M_1 = \begin{vmatrix} f(y_1 - y_1) & \cdots & f(y_1 - y_K) \\ \vdots & \ddots & \vdots \\ f(y_{K-1} - y_1) & \cdots & f(y_{K-1} - y_K) \\ F(y_{K-1} - y_1) - F(y_K - y_1) & \cdots & F(y_{K-1} - y_K) - \frac{1}{2} \end{vmatrix}. \tag{37}$$

Next, developing the determinant along the last row yields

$$\det M_1 = \textstyle\sum_{k=1}^{K} \{(-1)^{k+K} (F(y_{K-1} - y_k) - F(y_K - y_k)) \tag{38}$$
$$\times \begin{vmatrix} f(y_1 - y_1) & \cdots & f(y_1 - y_{k-1}) & f(y_1 - y_{k+1}) & \cdots & f(y_1 - y_K) \\ \vdots & & \vdots & \vdots & & \vdots \\ f(y_{K-1} - y_1) & \cdots & f(y_{K-1} - y_{k-1}) & f(y_{K-1} - y_{k+1}) & \cdots & f(y_{K-1} - y_K) \end{vmatrix} \}.$$

Using

$$F(y_{K-1} - y_k) - F(y_K - y_k) = \int_{y_K}^{y_{K-1}} f(t - y_k)dt \qquad (k \in \{1, \ldots, K\}), \tag{39}$$

this becomes

$$\det M_1 = \int_{y_K}^{y_{K-1}} \begin{vmatrix} f(y_1 - y_1) & \cdots & f(y_1 - y_K) \\ \vdots & \ddots & \vdots \\ f(y_{K-1} - y_1) & \cdots & f(y_{K-1} - y_K) \\ f(t - y_1) & \cdots & f(t - y_K) \end{vmatrix} dt. \tag{40}$$

As f is a proper Pólya frequency function, the determinant in (40) is seen to be positive for any $t \in (y_K, y_{K-1})$. Hence, $\det M_1 > 0$. In particular, M_1 is invertible, as claimed. It follows that there is at most one equilibrium in G_0. □

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