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A Note on Portfolio Selections under Various Risk Measures

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Abstract

This work gives a brief overview of the portfolio selection problem following the mean-risk approach first proposed by Markowitz (1952). We consider various risk measures, i.e. variance, value-at-risk and expected-shortfall and we study the efficient frontiers obtained by solving the portfolio selection problem under these measures. We show that under the assumption that returns are normally distributed, the efficient frontiers obtained by taking value-at-risk or expected-shortfall are subsets of the mean-variance efficient frontier. We generalize this result for all risk measures having the form $\gamma\sigma - \mu$ for some positive parameter γ, where μ is the mean and σ the variance and we show that for these measures Tobin separation holds under some restriction on the choice of γ.

Keywords: decision under risk, mean-risk models, portfolio optimization, value-at-risk, expected shortfall, efficient frontier.
JEL Classification: G11.

1 Introduction

The mean-risk approach for portfolio selection first proposed by Markowitz (1952) is very intuitive and, due to its simplicity, is also commonly used in practical financial decisions. In his seminal paper, Markowitz (1952) proposed the variance as measure of risk. The advantage of using the variance for describing the risk component of a portfolio, is principally due to the simplicity of the computation, but from the point of view of risk measurement the variance is not a satisfactory measure. First, the variance is a symmetric measure and "penalizes" gains and losses in the same way. Second, the variance is inappropriate to describe the risk of low probability events, as for example the default risk. Finally, mean-variance decisions are usually not consistent with the expected utility approach, unless returns are normally distributed or a quadratic utility index is chosen. We address our attention to this last point in an other work (see De Giorgi 2002).
As already suggested by Markowitz (1959), other risk measures can be used in the mean-risk

approach. In this work we consider value-at-risk (VaR) (Jorion 1997, Duffie and Pan 1997) and expected-shortfall (ES) (Acerbi, Nordio, and Sirtori 2001), which are quantile-based risk measures.

We show that under the assumption that return are normally distributed, the efficient frontiers resulting from the mean-VaR and from the mean-ES optimization are subsets of the mean-variance efficient frontier. The equivalence of these optimization problems under multivariate normal distribution has been first stated by Rockafellar and Uryasev (1999, Proposition 4.1). The equivalence of mean-ES and mean-variance analysis has been also proved by Hürlimann (2002), but for a more general class of distribution function for the returns, i.e. the class of elliptic distributions. In this work we restrict our attention to the multivariate Gaussian case and we give a precise statement on the efficient frontiers. We will see that although mean-variance, mean-VaR and mean-ES analysis are equivalent under multivariate Gaussian distributed returns, the mean-variance efficient portfolios can been inefficient under mean-ES or mean-VaR portfolio selection. Leippold (2001) (and for a more general framework, Leippold, Vanini, and Troiani (2002) and other authors cited in this last reference) has considered the impact of value-at-risk and expected-shortfall limits on the mean-variance portfolio allocation and has shown for multivariate Gaussian returns that VaR and ES constraints reduces the set of efficient portfolio allocations: this results will follows directly from our analysis. In the case that a risk-free asset is available, the set of efficient portfolios resulting from mean-VaR or from mean-ES portfolio selection are identical to the mean-variance efficient frontier, unless they are empty. This results allows an extension of the Tobin separation also for the case where investors use VaR or ES for describing their risk.

The paper is organized as follows. Section 2 introduces our notation and definitions. Section 3 consider value-at-risk and expected shortfall for normal distributed returns. Section 4 and Section 5 are devoted to the portfolio optimization problem without and with risk-free asset. Section 6 extends the optimization problem for expected shortfall to the case where returns are not necessarily normally distributed. We present an example based on the Swiss Market Index, where the mean-ES portfolio selection is identical to the mean-variance portfolio selection, in spite of the fact that we are not assuming that returns are multivariate distributed. This is not surprising, since the work of Hürlimann (2002). Finally, Section 7 concludes.

2   Notation and definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We suppose that K risky assets with return $R^1, \ldots, R^K$ respectively are available (random variables on $(\Omega, \mathcal{F}, \mathbb{P})$). With $\mathbf{R'_1} = (R^1, \ldots, R^K)$ we denote the vector of risky returns[1]. The risk-free asset, if it exists, is denoted by R^0 and $\mathbf{R'} = (R^0, \mathbf{R'_1})$. We write the boldfaced index $\mathbf{1}$ for K-dimensional vectors and no index for $(K+1)$-dimensional vectors. The last K components of a $(K+1)$-dimensional vector are also denoted with a boldfaced index $\mathbf{1}$ (this will be helpful when we write the optimization problem with the risk-free asset). Let $\Delta^{K-1} = \{\boldsymbol{w_1} \in \mathbb{R}^K \,|\, \sum_{k=1}^K w^k = 1\}$ denote the set of all portfolios of risky assets. Analogously, $\Delta^K = \{\boldsymbol{w} \in \mathbb{R}^{K+1} \,|\, \sum_{k=0}^K w^k = 1\}$ denotes the set of portfolio of risky assets and the risk-free asset. We sometime call the portfolio weights $\boldsymbol{\lambda_1}$ (or $\boldsymbol{\lambda}$) a *strategy*, since they fully characterize the portfolio decision of an investor. We do not impose short-sales constraints or diversification constraints. For an element $\boldsymbol{\lambda_1} \in \Delta^{K-1}$ (or $\boldsymbol{\lambda} \in \Delta^K$), the portfolio return is given by

$$R_{\boldsymbol{\lambda_1}} = \sum_{k=1}^K \lambda^k R^k = \mathbf{R'_1} \boldsymbol{\lambda_1} \quad (R_{\boldsymbol{\lambda}} = \mathbf{R'} \boldsymbol{\lambda}).$$

---

[1]The apostrophe near the vector means "transpose".

We assume that $\mathbb{P}\big[\omega \in \Omega \,|\, R^k(\omega) < R^0\big] > 0$ for all $k = 1, \ldots, K$, i.e. if the risk-free asset exists, risky assets have smaller return than the risk-free return with positive probability. This ensures that a risky asset does not dominate the risk-free asset.

Let $\mu^k = \mathbb{E}\big[R^k\big]$ denote the expected return of asset k for $k = 0, \ldots, K$. Obviously, $\mu^0 = R^0$. Moreover, for two risky assets R^j and R^k ($j \neq k$), $V^{jk} = Cov(R^j, R^k)$ denotes the covariance between asset j and asset k. For $j = k$ we have $V^{jk} = Var(R^j) = \sigma_j^2$ is the variance of asset j. $V = (V^{jk})_{1 \leq j,k \leq K}$ is the variance-covariance matrix. For a portfolio $\boldsymbol{\lambda_1} \in \Delta^{K-1}$ we have

$$\mathbb{E}\big[R_{\boldsymbol{\lambda_1}}\big] = \mu_{\boldsymbol{\lambda_1}} = \boldsymbol{\lambda_1'}\boldsymbol{\mu_1}, \tag{1}$$

$$Var(R_{\boldsymbol{\lambda_1}}) = \sigma_{\boldsymbol{\lambda_1}}^2 = \boldsymbol{\lambda_1'}V\boldsymbol{\lambda_1}. \tag{2}$$

Analogously for $\boldsymbol{\lambda} = (\lambda^0, \boldsymbol{\lambda_1}) \in \Delta^K$, $\mu_{\boldsymbol{\lambda}} = \boldsymbol{\lambda'}\boldsymbol{\mu} = \lambda^0\mu^0 + \boldsymbol{\lambda_1'}\boldsymbol{\mu_1}$ and $\sigma_{\boldsymbol{\lambda}}^2 = \sigma_{\boldsymbol{\lambda_1}}^2$.

We now introduce the definition of value-at-risk and expected-shortfall.

Definition 2.1 (Value-at-Risk). *Let $\alpha \in (0,1)$ be a given probability level and $\boldsymbol{\lambda_1} \in \Delta^{K-1}$. The* value-at-risk *at level α for the return $R_{\boldsymbol{\lambda_1}}$ is defined as*

$$VaR_\alpha(R_{\boldsymbol{\lambda_1}}) = -\inf\{x \,|\, \mathbb{P}\big[R_{\boldsymbol{\lambda_1}} \leq x\big] \geq \alpha\} = -F_{R_{\boldsymbol{\lambda_1}}}^{-1}(\alpha). \tag{3}$$

The function $F_{R_{\boldsymbol{\lambda_1}}}^{-1}$ (defined by the left hand side of the last equality sign) is called the generalized inverse *of the cumulative distribution function $F_{R_{\boldsymbol{\lambda_1}}}(x) = \mathbb{P}\big[R_{\boldsymbol{\lambda_1}} \leq x\big]$ of $R_{\boldsymbol{\lambda_1}}$ and gives the α-quantile of $R_{\boldsymbol{\lambda_1}}$.*

$VaR_\alpha(R_{\boldsymbol{\lambda_1}})$ is the maximal potential loss that portfolio $\boldsymbol{\lambda_1}$ can suffer in the $100(1-\alpha)\%$ best cases, i.e. with a small probability α the portfolio return is smaller than $-VaR_\alpha(R_{\boldsymbol{\lambda_1}})$. Therefore, for fixed α we would like to minimize the VaR_α over the set Δ^{K-1}.

Remark
Let $\boldsymbol{\lambda} \in \Delta^K$, $\boldsymbol{\lambda'} = (\lambda^0, \boldsymbol{\lambda_1'})$, then

$$\begin{aligned}
VaR_\alpha(R_{\boldsymbol{\lambda}}) &= -\inf\{x \,|\, \mathbb{P}\big[R_{\boldsymbol{\lambda}} \leq x\big] \geq \alpha\} \\
&= -\inf\{x \,|\, \mathbb{P}\big[R_{\boldsymbol{\lambda_1}} \geq x - \lambda^0\mu^0\big] \geq \alpha\} = VaR_\alpha(R_{\boldsymbol{\lambda_1}}) - \lambda^0\mu^0.
\end{aligned}$$

For general multivariate distribution functions for the returns $\mathbf{R_1}$, it is usually not possible to obtain a useful (i.e. an explicit function of the weights $\boldsymbol{\lambda_1}$) analytical expression for the VaR_α of a portfolio. The portfolio selection problem should be solved numerically and also the numerical approach is challenging since VaR_α is not a convex measure (see Gaivoronski and Pflug 2000, Vanini and Vignola 2001). The return of a portfolio is essentially given by a linear combination of K random variables and thus one should look at the multivariate distribution of the assets' returns and how the univariate distribution is affected by the weights of the portfolio. This is usually a non-trivial task. An alternative approach considers separately the univariate distributions of the single assets' returns and the dependence structure through the copula (see for example Embrechts, Höing, and Juri 2002, Juri 2002, Corbett and Rajaram 2002). A detailed introduction to the copula and its applications in finance can be found in Juri (2002) and Embrechts, Lindskog, and McNeil (2001). Here we just consider the simple case where returns are normally distributed, i.e. the multivariate distribution of the returns $\mathbf{R_1}$ is a Gaussian multivariate distribution with mean $\boldsymbol{\mu_1}$ and variance-covariance matrix V. This assumption strongly simplifies the analysis, as we will see in the next section.

We now proceed with the definition of expected shortfall.

Definition 2.2 (Expected-Shortfall). *Let $\alpha \in (0, 1)$ be a given probability level and $\boldsymbol{\lambda_1} \in \Delta^{K-1}$. The* expected-shortfall *at level α for the return $R_{\boldsymbol{\lambda_1}}$ is defined as*

$$ES_\alpha(R_{\boldsymbol{\lambda_1}}) = -\frac{1}{\alpha}\left(\mathbb{E}\big[R_{\boldsymbol{\lambda_1}}1_{\{R_{\boldsymbol{\lambda_1}} \le x^{(\alpha)}\}}\big] - x^{(\alpha)}\left(\mathbb{P}\big[R_{\boldsymbol{\lambda_1}} \le x^{(\alpha)}\big] - \alpha\right)\right), \tag{4}$$

where $x^{(\alpha)} = F_{R_{\boldsymbol{\lambda_1}}}^{-1}(\alpha)$.

Acerbi and Tasche (2001a) shown that ES_α is the limit (in probability) of a natural estimator for the expected losses in the $100\alpha\%$ worst cases, i.e. the average over the $100\alpha\%$ worst outcomes, multiplied by -1. Moreover they shown that ES_α is a coherent risk measure in the sense of Artzner, Delbaen, Eber, and Heath (1999). There is some confusion in the literature about the nomenclatures of the various quantile-based risk measures. Some authors identify the expected-shortfall with the tail conditional expectation (Pflug 2000), other authors use the terminology conditional value-at-risk (Rockafellar and Uryasev 2001) for the expected shortfall just introduced. We give the definition of tail conditional expectation (which differs from the definition of ES already formulated) and we show that this measure is identical to the expected shortfall only under some conditions.

Definition 2.3 (Tail Conditional Expectation). *Let $\alpha \in (0, 1)$ be a given probability level and $\boldsymbol{\lambda_1} \in \Delta^{K-1}$. The* tail conditional expectation *at level α for the return $R_{\boldsymbol{\lambda_1}}$ is defined as*

$$TCE_\alpha(R_{\boldsymbol{\lambda_1}}) = -\mathbb{E}\big[R_{\boldsymbol{\lambda_1}} \,|\, R_{\boldsymbol{\lambda_1}} \le x^{(\alpha)}\big], \tag{5}$$

where $x^{(\alpha)} = F_{R_{\boldsymbol{\lambda_1}}}^{-1}(\alpha)$.

The tail conditional expectation is a coherent risk measure only for continuous distribution functions. In fact it may violate the subadditivity for general distributions (see Acerbi and Tasche 2001b). Moreover, as pointed out by Acerbi and Tasche (2001a), TCE does not answer the question about the expected loss incurred in the $100\alpha\%$ worst cases, since the set $\{R_{\boldsymbol{\lambda_1}} \le x^{(\alpha)}\}$ could have a probability larger than α if the distribution function is not continuous. On the other side, for continuous distribution functions, we have $\mathbb{P}\big[R_{\boldsymbol{\lambda_1}} \le x^{(\alpha)}\big] = \alpha$ and thus equation (4) implies $ES_\alpha = TCE_\alpha$. This last equation explains why some authors refer to the expected shortfall with the terminology "conditional value-at-risk".

In general we have the following results (Pflug 2000):

$$ES_\alpha(R_{\boldsymbol{\lambda_1}}) = TCE_\alpha(R_{\boldsymbol{\lambda_1}}) + (\beta - 1)\left(TCE_\alpha(R_{\boldsymbol{\lambda_1}}) - VaR_\alpha(R_{\boldsymbol{\lambda_1}})\right),$$

where $\beta = \frac{1}{\alpha}\mathbb{P}\big[R_{\boldsymbol{\lambda_1}} \le -VaR_\alpha(R_{\boldsymbol{\lambda_1}})\big]$.

3 VaR and ES with normal distributed returns

In this section we make a strong assumption on the multivariate distribution of the return vector $\mathbf{R_1}$. We assume that $\mathbf{R_1} \sim \mathcal{N}(\boldsymbol{\mu_1}, V)$, i.e. $\mathbf{R_1}$ is multivariate Gaussian distributed with mean $\boldsymbol{\mu_1}$ and variance-covariance matrix V. We have the following result for value-at-risk and expected shortfall

Lemma 3.1. *Let $\alpha \in (0, 1)$ and $\mathbf{R_1} \sim \mathcal{N}(\boldsymbol{\mu_1}, V)$. Then $R_{\boldsymbol{\lambda_1}} \sim \mathcal{N}(\mu_{\boldsymbol{\lambda_1}}, \sigma_{\boldsymbol{\lambda_1}}^2)$ and*

(i)

$$VaR_\alpha(R_{\boldsymbol{\lambda_1}}) = z^{(\alpha)}\sigma_{\boldsymbol{\lambda_1}} - \mu_{\boldsymbol{\lambda_1}}, \tag{6}$$

where $z^{(\alpha)} = \Phi^{-1}(1 - \alpha)$, $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}z^2}\,dz$.

(ii)

$$ES_\alpha(R_{\boldsymbol{\lambda_1}}) = \frac{\rho(z^{(\alpha)})}{\alpha}\sigma_{\boldsymbol{\lambda_1}} - \mu_{\boldsymbol{\lambda_1}}, \qquad (7)$$

where $\rho(\cdot)$ is the density of the standard normal distribution, $\rho(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$.

Proof. (i) Since $R_{\boldsymbol{\lambda_1}} \sim N(\mu_{\boldsymbol{\lambda_1}}, \sigma^2_{\boldsymbol{\lambda_1}})$, then $\frac{R_{\boldsymbol{\lambda_1}} - \mu_{\boldsymbol{\lambda_1}}}{\sigma_{\boldsymbol{\lambda_1}}} \sim N(0,1)$. Therefore we have:

$$
\begin{aligned}
\inf\{x\,|\,\mathbb{P}\left[R_{\boldsymbol{\lambda_1}} \geq x\right] \geq \alpha\} &= \sigma_{\boldsymbol{\lambda_1}}\inf\{x\,|\,\mathbb{P}\left[\frac{R_{\boldsymbol{\lambda_1}} - \mu_{\boldsymbol{\lambda_1}}}{\sigma_{\boldsymbol{\lambda_1}}} \leq \frac{x - \mu_{\boldsymbol{\lambda_1}}}{\sigma_{\boldsymbol{\lambda_1}}}\right] \geq \alpha\} + \mu_{\boldsymbol{\lambda_1}} \\
&= \sigma_{\boldsymbol{\lambda_1}}\Phi^{-1}(\alpha) + \mu_{\boldsymbol{\lambda_1}} = -\sigma_{\boldsymbol{\lambda_1}}\Phi^{-1}(1-\alpha) + \mu_{\boldsymbol{\lambda_1}} \\
&= -\sigma_{\boldsymbol{\lambda_1}}z^{(\alpha)} + \mu_{\boldsymbol{\lambda_1}}.
\end{aligned}
$$

Take the minus sign, and the first statement of the Lemma follows.

(ii) Since the normal distribution is continuous, we can calculate the ES_α as TCE_α. From the first part of this proof we know that $VaR_\alpha(R_{\boldsymbol{\lambda_1}}) = z^{(\alpha)}\sigma_{\boldsymbol{\lambda_1}} - \mu_{\boldsymbol{\lambda_1}}$. We obtain

$$
\begin{aligned}
\mathbb{P}\left[R_{\boldsymbol{\lambda_1}} \leq x\,|\,R_{\boldsymbol{\lambda_1}} \leq -VaR_\alpha(R_{\boldsymbol{\lambda_1}})\right] &= \frac{1}{\alpha}\mathbb{P}\left[R_{\boldsymbol{\lambda_1}} \leq x\right]1_{\{x \leq -VaR_\alpha(R_{\boldsymbol{\lambda_1}})\}} \\
&= \frac{1}{\alpha}\mathbb{P}\left[R_{\boldsymbol{\lambda_1}} \leq x\right]1_{\{x \leq -z^{(\alpha)}\sigma_{\boldsymbol{\lambda_1}} + \mu_{\boldsymbol{\lambda_1}}\}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathbb{E}\left[R_{\boldsymbol{\lambda_1}}\,|\,R_{\boldsymbol{\lambda_1}} \leq -VaR_\alpha(R_{\boldsymbol{\lambda_1}})\right] &= \\
&= \frac{1}{\alpha}\int_{-\infty}^{-z^{(\alpha)}\sigma_{\boldsymbol{\lambda_1}} + \mu_{\boldsymbol{\lambda_1}}} z\frac{1}{\sqrt{2\pi}\sigma_{\boldsymbol{\lambda_1}}}\exp\left(-\frac{1}{2}\left(\frac{z - \mu_{\boldsymbol{\lambda_1}}}{\sigma_{\boldsymbol{\lambda_1}}}\right)^2\right) dz \\
&= \frac{1}{\alpha}\int_{-\infty}^{-z^{(\alpha)}} \frac{1}{\sqrt{2\pi}}(\sigma_{\boldsymbol{\lambda_1}}u + \mu_{\boldsymbol{\lambda_1}})\exp\left(-\frac{1}{2}u^2\right) du \\
&= \frac{1}{\alpha\sqrt{2\pi}}\left[\sigma_{\boldsymbol{\lambda_1}}\int_{-\infty}^{-z^{(\alpha)}} u\exp\left(-\frac{1}{2}u^2\right) du + \mu_{\boldsymbol{\lambda_1}}\int_{-\infty}^{-z^{(\alpha)}} \exp\left(-\frac{1}{2}u^2\right) du\right] \\
&= \frac{1}{\alpha}\left[-\rho(-z^{(\alpha)})\sigma_{\boldsymbol{\lambda_1}} + \Phi(-z^{(\alpha)})\mu_{\boldsymbol{\lambda_1}}\right] = \frac{1}{\alpha}\left[-\rho(z^{(\alpha)})\sigma_{\boldsymbol{\lambda_1}} + \underbrace{(1 - \Phi(z^{(\alpha)}))}_{=\alpha}\mu_{\boldsymbol{\lambda_1}}\right] \\
&= -\frac{\rho(z^{(\alpha)})}{\alpha}\sigma_{\boldsymbol{\lambda_1}} + \mu_{\boldsymbol{\lambda_1}}.
\end{aligned}
$$

Take the minus sign and the second statement of the Lemma follows.

$\square$

The Lemma shows that under the assumption of normal distributed returns, value-at-risk and expected shortfall can be fully characterized by the mean and the variance of the portfolio, i.e. we have the following general form

$$risk(R_{\boldsymbol{\lambda_1}}) = \gamma(\alpha)\sigma_{\boldsymbol{\lambda_1}} - \mu_{\boldsymbol{\lambda_1}}, \qquad (8)$$

where $\gamma(\alpha)$ is a function of the level α taking the value $z^{(\alpha)}$ for value-at-risk and $\frac{\rho(z^{(\alpha)})}{\alpha}$ for expected shortfall. This is naturally a very special case due to the special distribution. For a general *continuous* distribution function we obtain the following representation for ES_α and VaR_α (Rockafellar and Uryasev 1999):

Lemma 3.2. *Let $R_{\boldsymbol{\lambda_1}}$ be the return of the portfolio $\boldsymbol{\lambda_1} \in \Delta^{K-1}$ and suppose that $R_{\boldsymbol{\lambda_1}}$ has a continuous cumulative distribution function. Then*

$$ES_\alpha(R_{\boldsymbol{\lambda_1}}) = \inf\{a + \frac{1}{\alpha}\mathbb{E}\big[(-R_{\boldsymbol{\lambda_1}} - a)^+\big]\,|\,a \in \mathbb{R}\} \tag{9}$$

and

$$VaR_\alpha(R_{\boldsymbol{\lambda_1}}) \in \arg\inf\{a + \frac{1}{\alpha}\mathbb{E}\big[(-R_{\boldsymbol{\lambda_1}} - a)^+\big]\,|\,a \in \mathbb{R}\}, \tag{10}$$

where $x^+ = \max(x, 0)$.

The Lemma already suggest the difficulty to solve the optimization problem for VaR_α under general continuous distribution functions, since in fact the problem is usually not a convex one and the solution may be not unique.

4 Portfolio optimization and efficient frontiers

The mean-risk approach for portfolio selection essentially consists in minimizing the *risk* of the portfolio return over the set of strategies, given a fixed expected return that must be reached. Mathematically the portfolio optimization can be written as the following problem $(\mathcal{M}_{risk})$[2]: find $\boldsymbol{w}^* \in \Delta$ that solves

$$\min_{\boldsymbol{w} \in \Delta} risk(R_{\boldsymbol{w}})$$
$$\text{s.t.} \quad \mathbb{E}\big[R_{\boldsymbol{w}}\big] = \overline{\mu}$$

where Δ is equal Δ^{K-1} if no risk-free asset is available, or Δ^K if the risk-free asset exists. We call the portfolio $\boldsymbol{w}^*$ an *optimal portfolio* for the expected return $\overline{\mu}$: we will see later that an optimal portfolio may be inefficient. We denote by $\mathcal{B}_{(\mu,risk)}$ the *$(\mu, risk)$-boundary*, i.e. the subset of $\mathbb{R}^2$ containing all pairs $(risk(R_{\boldsymbol{w}^*}), \overline{\mu})$ where $\boldsymbol{w}^*$ is the optimal portfolio for the expected return $\overline{\mu}$. Here, the function $risk(\cdot)$ can take the following forms:

  (i) $risk(R_{\boldsymbol{w}}) = \sigma_{\boldsymbol{w}}$;

  (ii) $risk(R_{\boldsymbol{w}}) = VaR_\alpha(R_{\boldsymbol{w}})$ for some $\alpha \in (0, 1)$,

  (iii) $risk(R_{\boldsymbol{w}}) = ES_\alpha(R_{\boldsymbol{w}})$ for some $\alpha \in (0, 1)$,

where $\boldsymbol{w}$ denotes a strategy in Δ. From the previous section we know that when returns are multivariate Gaussian distributed, then both the $risk(\cdot)$ functions in (ii) and (iii) have the form $\gamma(\alpha)\sigma_{\boldsymbol{w}} - \mu_{\boldsymbol{w}}$ for some fixed parameter $\gamma(\alpha)$ depending on the choice of the level α. We additionally suppose that $\alpha \in (0, 0.5)$ (usually α take value smaller than 0.1) so that we can assume that $\gamma(\alpha) > 0$ for both value-at-risk and expected shortfall. When the $risk(\cdot)$ function has the form $risk^\gamma(R_{\boldsymbol{w}}) = \gamma\sigma_{\boldsymbol{w}} - \mu_{\boldsymbol{w}}$ for some $\gamma > 0$, then the optimization problem $(\mathcal{M}_{risk})$ has obviously the same solution on Δ as the analogous optimization problem with the standard deviation as measure of risk (we denote this problem by $(\mathcal{M}_\sigma)$), but naturally the corresponding minimal values for the risk functions are different. Equivalence of the optimization problems, concerns the mathematical solutions. Due to the analogy of the optimization problems $(\mathcal{M}_{risk})$ and $(\mathcal{M}_\sigma)$, with $risk$ some measure of the form $\gamma\sigma_{\boldsymbol{w}} - \mu_{\boldsymbol{w}}$, $\gamma > 0$, we first give a characterization of the solutions of $(\mathcal{M}_\sigma)$ and we then extend some of the results to the case $(\mathcal{M}_{risk})$.

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[2]A remark on the notation. In this paper we use the italic letter $\mathcal{M}$ for denoting the Markowitz problem. The risk measure is expressed in the index of the $\mathcal{M}$.

4.1 Mean-variance portfolio optimization

We now consider the problem $(\mathcal{M}_\sigma)$ for the case that the risk-free asset is not available. We have the following optimization problem

$$\min_{\boldsymbol{\lambda_1} \in \Delta^{K-1}} \sigma_{\boldsymbol{\lambda_1}}$$
$$\text{s.t.} \quad \mu_{\boldsymbol{\lambda_1}} = \overline{\mu}.$$

From equations (1) and (2) we obtain a quadratic objective function with linear contraints. Under the assumption that V (the variance-covariance matrix) is strictly positive definite, that the vectors $\boldsymbol{\mu_1}$ and $\mathbf{e_1} = (1, \ldots, 1)' \in \mathbb{R}^K$ are linearly independent and that all the first and second moments exist, a portfolio $\boldsymbol{\lambda_1^*}(\overline{\mu}) \in \Delta^{K-1}$ solves the optimization problem $(\mathcal{M}_\sigma)$ if and only if:

$$\boldsymbol{\lambda_1^*}(\overline{\mu}) = \overline{\mu}\,\boldsymbol{\lambda_1^{*,0}} - \boldsymbol{\lambda_1^{*,1}} \tag{11}$$

where

$$
\begin{aligned}
\boldsymbol{\lambda_1^{*,0}} &= \frac{1}{D}(BV^{-1}\boldsymbol{\mu_1} - CV^{-1}\boldsymbol{e_1}), \\
\boldsymbol{\lambda_1^{*,1}} &= \frac{1}{D}(CV^{-1}\boldsymbol{\mu_1} - AV^{-1}\boldsymbol{e_1}),
\end{aligned}
$$

and

$$
\begin{aligned}
A &= \mu_1' V^{-1}\mu_1, \\
B &= e_1' V^{-1}e_1, \\
C &= e_1' V^{-1}\mu_1, \\
D &= AB - C^2.
\end{aligned}
$$

This result follows directly from the first order conditions. Note that $D > 0$ by the Cauchy-Schwarz inequality. The parameter $\overline{\mu}$ enters in the characterization of the optimal portfolio only in equation (11). Equation (11) can be viewed as a "mutual fund" representation, since in fact it states that homogeneous investors[3] still choose a combination of $\boldsymbol{\lambda_1^{*,0}}$ and $\boldsymbol{\lambda_1^{*,1}}$, where the proportion invested in $\boldsymbol{\lambda_1^{*,0}}$ depends on the target expectation $\overline{\mu}$. Moreover, from equation (11) one can find the relation between $\overline{\mu}$ and the optimal standard deviation $\sigma_{\boldsymbol{\lambda_1^*}(\overline{\mu})}$. This gives us the mean-variance boundary $\mathcal{B}_{(\mu,\sigma)}$. Note that by equation (11), there is a one-to-one correspondence between $\mathcal{B}_{(\mu,\sigma)}$ and the subset of optimal portfolio in Δ^{K-1} under the assumptions on V and $\boldsymbol{\mu_1}$ given above. We have

$$(\sigma, \overline{\mu}) \in \mathcal{B}_{(\mu,\sigma)} \iff \frac{\sigma^2}{1/B} - \frac{(\overline{\mu} - C/B)^2}{D/B^2} = 1. \tag{12}$$

The right hand side of equation (12) defines an hyperbole in $\mathbb{R}^+ \times \mathbb{R}$.

Definition 4.1 (Global minimum *risk* portfolio). *A portfolio $\boldsymbol{w} \in \Delta$ is a* global minimum *risk* portfolio *in Δ, if and only if*

$$\boldsymbol{w} \in \arg\min \left\{ risk(R_{\boldsymbol{w}}) \,|\, (risk(R_{\boldsymbol{w}}), \mu_{\boldsymbol{w}}) \in \mathcal{B}_{(\mu,risk)} \right\}.$$

We denote the global minimum risk portfolio by $\mathbf{m}^{risk}$.

---

[3] "Homogeneous investors" means investors that have the same beliefs about the probability $\mathbb{P}$.

From equation (12) it follows that the global minimum variance on the mean-variance boundary can be obtained with $\overline{\mu} = \frac{C}{B}$. The corresponding optimal portfolio is the global minimum variance portfolio and is given by equation (11), we have

$$\mathbf{m_1^{\sigma}} = \frac{1}{C} V^{-1} \boldsymbol{\mu_1}. \tag{13}$$

Moreover, for $\overline{\mu} < \frac{C}{B}$ the corresponding optimal portfolio $\boldsymbol{\lambda_1^*}(\overline{\mu})$ given by equation (11) has the same variance has the portfolio $\boldsymbol{\lambda_1^*}(2\frac{C}{B} - \overline{\mu})$, but the latter provides an higher expected return, since $2\frac{C}{B} - \overline{\mu} > \overline{\mu}$. Thus optimal portfolios $\boldsymbol{\lambda_1^*}(\overline{\mu})$ for $\overline{\mu} < \frac{C}{B}$ will never be selected by a rational investor, i.e. they are not efficient. This motivate the following definition.

Definition 4.2 (($\mu, risk$)-efficient portfolio, ($\mu, risk$)-efficient frontier). *A portfolio $\boldsymbol{w} \in \Delta$ is ($\mu, risk$)-efficient in Δ, if and only if no portfolio $\boldsymbol{v} \in \Delta$ exists such that $\mu_{\boldsymbol{v}} \geq \mu_{\boldsymbol{w}}$ and $risk(R_{\boldsymbol{v}}) \leq risk(R_{\boldsymbol{w}})$ where at least one of the inequalities is strict. Δ denotes as before the set Δ^{K-1} if no risk-free asset is available, and Δ^K else. The ($\mu, risk$)-efficient frontier, denoted by $\mathcal{E}_{(\mu, risk)}$, is the subset of $\mathbb{R}^2$ defined by*

$$\mathcal{E}_{(\mu, risk)} = \{(risk(R_{\boldsymbol{w}}), \mu_{\boldsymbol{w}}) \in \mathbb{R}^2 \,|\, \boldsymbol{w} \in \Delta \text{ is } (\mu, risk)\text{-efficient}\}.$$

Remark
A necessary condition for a portfolio to be efficient is that it is optimal for some $\overline{\mu}$. As we have already shown for the (μ, σ) portfolio optimization, an optimal portfolio could be inefficient. For the (μ, σ) optimization, there is a one-to-one correspondence between the set of (μ, σ)-efficient portfolios and the (μ, σ)-efficient frontier.

The following Proposition characterize the (μ, σ)-efficient frontier.

Proposition 4.1.

$$\mathcal{E}_{(\mu, \sigma)} = \left\{ (\sigma, \mu) \in \mathcal{B}_{(\mu, \sigma)} \,\Big|\, \mu \geq \frac{C}{B} \right\}. \tag{14}$$

Proof. Follows directly from equation (12) and the Definition 4.2. $\square$

4.2 Mean-$risk^{\gamma}$ portfolio optimization

In this subsection we consider the following problem ($\mathcal{M}_{risk^{\gamma}}$):

$$\min_{\boldsymbol{\lambda_1} \in \Delta^{K-1}} risk^{\gamma}(R_{\boldsymbol{\lambda_1}})$$
$$\text{s.t.} \quad \mu_{\boldsymbol{\lambda_1}} = \overline{\mu},$$

where $risk^{\gamma}(R_{\boldsymbol{\lambda_1}}) = \gamma \sigma_{\boldsymbol{\lambda_1}} - \mu_{\boldsymbol{\lambda_1}}$, $\gamma > 0$. It follows

$$\mathcal{B}_{(\mu, risk^{\gamma})} = \left\{ (risk^{\gamma}, \mu) \in \mathbb{R}^2 \,\Big|\, (\frac{risk^{\gamma} + \mu}{\gamma}, \mu) \in \mathcal{B}_{(\mu, \sigma)} \right\}. \tag{15}$$

Thus the set of (μ, σ) optimal portfolios remains unchanged under the $(\mu, risk^{\gamma})$ portfolio decision. From equation (15) we obtain the following Proposition.

Proposition 4.2. *The global minimum $risk^{\gamma}$ portfolio exists if and only if $\gamma > \sqrt{\frac{D}{B}}$. In this case it is given by*

$$\boldsymbol{m_1^{risk}} = \overline{\mu}^{\min}(\gamma) \boldsymbol{\lambda_1^{*,0}} - \boldsymbol{\lambda_1^{*,1}}, \tag{16}$$

where $\overline{\mu}^{\min}(\gamma) = \frac{C}{B} + \sqrt{\frac{D}{B}\left(\frac{\gamma^2}{B\gamma^2 - D} - \frac{1}{B}\right)}$.

Proof. Generalization of Proposition 1 in Alexandre and Baptista (2000). □

Corollary 4.1. *The global minimum $risk^\gamma$ portfolio, if it exists, is also (μ, σ)-efficient.*

Proof. From equation (14) since obviously $\overline{\mu}^{\min}(\gamma) > \frac{C}{B}$ for $\gamma > \sqrt{\frac{D}{B}}$. □

Corollary 4.2. *Suppose the $\boldsymbol{R_1} \sim \mathcal{N}(\boldsymbol{\mu_1}, V)$, then*

  (i) *the global minimum VaR_α portfolio exists if and only if $\alpha < 1 - \Phi\left(\sqrt{\frac{D}{B}}\right)$;*

  (ii) *the global minimum ES_α portfolio exists if and only if $\frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})} > \sqrt{\frac{D}{B}}$.*

Proof. Under the assumption of multivariate normally distributed returns we have $VaR_\alpha = risk^\gamma$ with $\gamma = z^{(\alpha)} = \Phi^{-1}(1 - \alpha)$ and $ES_\alpha = risk^\gamma$ with $\gamma = \frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})}$. □

Remark
The equation $\frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})} = \sqrt{\frac{D}{B}}$ cannot be solved explicitly for $z^{(\alpha)}$. Nevertheless, we observe that the function $\frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})}$ is increasing in $z^{(\alpha)}$ on $\mathbb{R}^+$ and thus decreasing in α on $(0, 0.5)$. Moreover, for $\alpha \nearrow 0.5$, $\frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})} \searrow \frac{2}{\sqrt{2\pi}}$ and for $\alpha \searrow 0$, $\frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})} \nearrow \infty$. Thus, we should still be able to find a maximal level β such that for $\alpha \in (0, \beta)$ the inequality $\frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})} > \sqrt{\frac{D}{B}}$ holds. Finally, since $\frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})} > z^{(\alpha)}$ for $\alpha \in (0.0.5)$, the maximal level $\beta \in (0, 0.5)$ for which $\frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})} > \sqrt{\frac{D}{B}}$ holds in $(0, \beta)$, is greater than the level $1 - \Phi\left(\sqrt{\frac{D}{B}}\right)$ from the previous Corollary for VaR.

Corollary 4.3. *Let $\boldsymbol{\lambda_1} \in \Delta^{K-1}$ be a (μ, σ)-efficient portfolio and suppose that $\mu_{\boldsymbol{\lambda_1}} > \frac{C}{B}$, then*

$$\boldsymbol{\lambda_1} = \boldsymbol{m_1}^{risk^\gamma}$$

for

$$\gamma = \sqrt{\frac{D}{B} + \frac{D^2/B^3}{(\mu_{\boldsymbol{\lambda_1}} - C/B)^2}}.$$

Proof. Solve the equation

$$\mu_{\boldsymbol{\lambda_1}} = \overline{\mu}^{\min}(\gamma) = \frac{C}{B} + \sqrt{\frac{D}{B}\left(\frac{\gamma^2}{B\gamma^2 - D} - \frac{1}{B}\right)}$$

for γ. Proposition 4.2 ensure that the portfolio defined by equation (16) is the global minimum $risk^\gamma$ portfolio with return $\mu_{\boldsymbol{\lambda_1}}$. □

The last Corollary suggests that under the assumption of multivariate distribution for assets' returns, one can find, for every (μ, σ)-efficient portfolio $\boldsymbol{\lambda_1}$ which differs from the global minimum variance portfolio $\boldsymbol{m_1^\sigma}$, a level α such that this portfolio corresponds to the global minimum VaR_α portfolio. The same holds for ES_α and a different level α. In the case of VaR_α, the Corollary also suggests that for levels of α converging to $1 - \Phi(\frac{D}{B})$ from below, the variance of the minimum global VaR_α portfolio converges to infinity.

Example

We consider two assets R^k (k=1,2) with $Var(R^1) = 0.02$, $Var(R^2) = 0.01$ and $corr(R^1, R^2) = 0.9$. Then $\frac{D}{B} = 4.951415$ and thus $1 - \Phi\left(\sqrt{\frac{D}{B}}\right) = 0.01303466$. It follows that the global minimum VaR_α portfolio exists only for $\alpha < 0.01303466$.

Moreover, for $\alpha = 0.013$ we have that $Var(R_{\mathbf{m_1}^{VaR}}) \cong 9$, i.e. the variance of the global minimum $VaR_{1.3\%}$ portfolio is quite large.

We now address the question whether the set of (μ, σ)-efficient portfolios and the (μ, σ)-efficient frontier also remain unchanged under the $(\mu, risk^\gamma)$ portfolio optimization. We have already seen, that the global minimum $risk^\gamma$ portfolio is (μ, σ)-efficient, if it exists. The following Proposition characterizes the $(\mu, risk^\gamma)$-efficient frontier.

Proposition 4.3. *(i) If $\gamma > \sqrt{\frac{D}{B}}$, then*

$$\mathcal{E}_{(\mu, risk^\gamma)} = \left\{ (risk^\gamma, \mu) \in \mathcal{B}_{(\mu, risk^\gamma)} \,\Big|\, \mu \geq \frac{C}{B} + \sqrt{\frac{D}{B}\left(\frac{\gamma^2}{B\gamma^2 - D} - \frac{1}{B}\right)} \right\}. \quad (17)$$

Moreover, every $(\mu, risk^\gamma)$-efficient portfolio is (μ, σ)-efficient, but the opposite is not true. A (μ, σ)-efficient portfolio $\boldsymbol{\lambda_1}$ is $(\mu, risk^\gamma)$-efficient if and only if $\mu_{\boldsymbol{\lambda_1}} \geq \frac{C}{B} + \sqrt{\frac{D}{B}\left(\frac{\gamma^2}{B\gamma^2 - D} - \frac{1}{B}\right)}$. Particularly, the global minimum variance portfolio is never $(\mu, risk^\gamma)$-efficient.

(ii) If $\gamma \leq \sqrt{\frac{D}{B}}$, then

$$\mathcal{E}_{(\mu, risk^\gamma)} = \emptyset. \quad (18)$$

Remark (i) If γ is small, then the risk measure $risk^\gamma$ is dominated by μ. In this case, since short sale is allowed, one could still reduce the risk by taking still more long positions on assets with high expected return and still more short positions on assets with low expected return. This operation simultaneously increases the expected return of the portfolio and thus no $(\mu, risk^\gamma)$-efficient portfolio will be found. In the extreme case that $\gamma = 0$, this is obvious from the fact that the $(\mu, risk^\gamma)$-boundary is a straight line with slope -1 (see Figure 1).

(ii) For $\gamma^1 > \gamma^2$, then $\overline{\mu}^{\min}(\gamma^1) < \overline{\mu}^{\min}(\gamma^2)$ and thus the set of $(\mu, risk^{\gamma^2})$-efficient portfolios is strictly contained in the set of $(\mu, risk^{\gamma^1})$-efficient portfolios.

Corollary 4.4. *Suppose the $\boldsymbol{R_1} \sim \mathcal{N}(\boldsymbol{\mu_1}, V)$, then*

(i) if $\alpha < 1 - \Phi\left(\sqrt{\frac{D}{B}}\right)$

$$\mathcal{E}_{(\mu, VaR_\alpha)} = \left\{ (VaR_\alpha, \mu) \in \mathcal{B}_{(\mu, VaR_\alpha)} \,\Big|\, \mu \geq \frac{C}{B} + \sqrt{\frac{D}{B}\left(\frac{(z^{(\alpha)})^2}{B(z^{(\alpha)})^2 - D} - \frac{1}{B}\right)} \right\}, \quad (19)$$

else if $\alpha < 1 - \Phi\left(\sqrt{\frac{D}{B}}\right)$, $\mathcal{E}_{(\mu, VaR_\alpha)} = \emptyset$.

(ii) if $\frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})} > \sqrt{\frac{D}{B}}$

$$\mathcal{E}_{(ES_\alpha, \mu)} = \left\{ (ES_\alpha, \mu) \in \mathcal{B}_{(\mu, ES_\alpha)} \,\Big|\, \mu \geq \frac{C}{B} + \sqrt{\frac{D}{B}\left(\frac{(\rho(z^{(\alpha)})/\alpha)^2}{B(\rho(z^{(\alpha)})/\alpha)^2 - D} - \frac{1}{B}\right)} \right\}, \quad (20)$$

else if $\frac{\rho(z^{(\alpha)})}{1 - \Phi(z^{(\alpha)})} \leq \sqrt{\frac{D}{B}}$, $\mathcal{E}_{(\mu, ES_\alpha)} = \emptyset$.
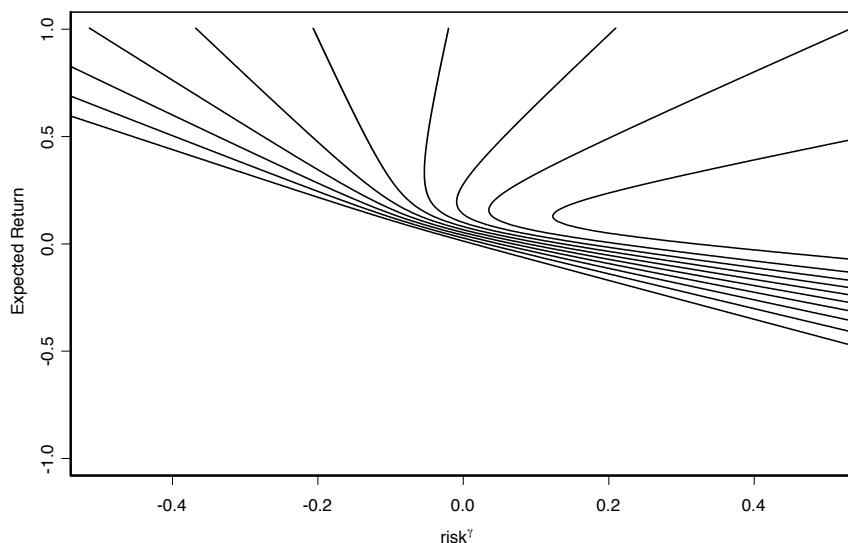
Figure 1: $(\mu, risk^\gamma)$-boundary for various values of γ. For decreasing γ, the $(\mu, risk^\gamma)$-boundary approaches to a straight line with slope -1. If $\gamma \leq \sqrt{\frac{D}{B}}$, than the $(\mu, risk^\gamma)$-boundary degenerates since the contribution of μ in the risk measure dominates. In this case, the $(\mu, risk^\gamma)$-boundary suggests that one could infinitely increase the expected return and decrease the $risk^\gamma$. Thus, no portfolio will be efficient.

Example (Continued)
With the same assets as in the previous example, the mean-VaR_α efficient frontier is empty for $\alpha > 0.01303466$. The mean-ES_α efficient frontier is empty for $\alpha > 0.03347571$.

Remark
Since $\frac{\rho(z^{(\alpha)})}{\alpha} \geq z^{(\alpha)}$ for $\alpha \in (0, 0.5)$, then by Remark 4.2 (ii) the set of (μ, VaR_α)-efficient portfolios is a strictly subset of the set of (μ, ES_α)-efficient portfolios.

Figure 2 gives an overview of the results from Proposition 4.3 and the Corollary.

Remark
The impact of VaR_α or ES_α constraints for (μ, σ) investors, can be analyzed directly by considering the analogies stated in the previous Corollary. Let us consider VaR. First, we observe that if for some $\alpha \in (0, 1)$ the constraint $\overline{VaR_\alpha}$ is smaller than $VaR(\mathbf{m}^{VaR_\alpha})$, then no optimal portfolio allocation could exists for (μ, σ) investors (and also for (μ, VaR_α) investors).
Thus, we assume that $\overline{VaR_\alpha} \geq VaR(\mathbf{m}^{VaR_\alpha})$. In this case the restricted mean-VaR_α boundary can be easily computed and is equal to $\mathcal{B}_{(\mu, VaR_\alpha)} \cap \{(x, y) \in \mathbb{R}^2 \,|\, x \leq \overline{VaR_\alpha}\}$. The restricted (μ, σ) boundary follows from equation (15). As already stated by Leippold (2001), two kind of (μ, σ) investors could be affected by the VaR_α restriction: investors who select high variance (μ, σ)-efficient portfolios and investors who select low variance (μ, σ)-efficient portfolios, which are not (μ, VaR_α) efficient. The first type of investors are asked to reduce the variance, in order to satisfy the VaR_α restriction. The second type of investors instead, are asked to select a portfolio with higher variance (see Figure 3).
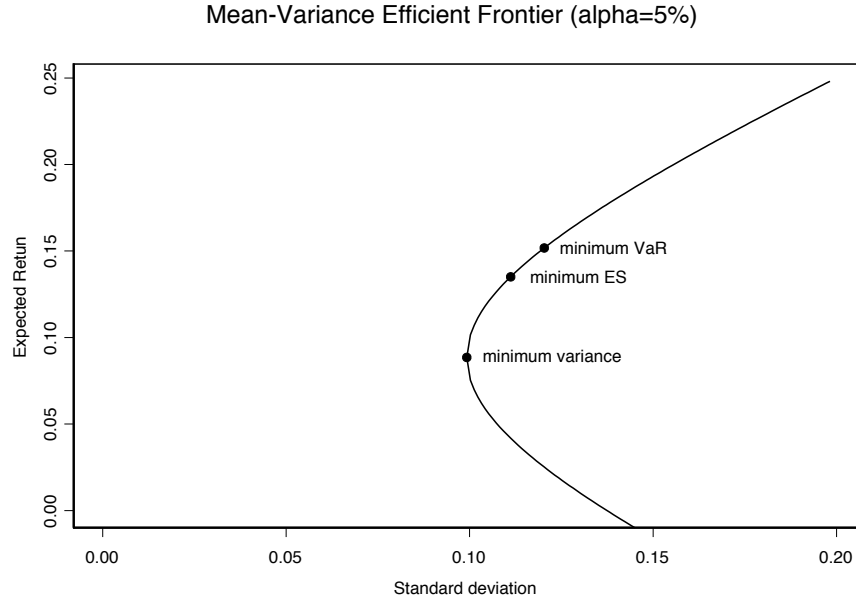
Mean-Variance Efficient Frontier (alpha=5%)

Figure 2: (μ, σ)-boundary with the global minimum variance portfolio, the global minimum $VaR_{5\%}$ portfolio and the global minimum $ES_{5\%}$ portfolio. The efficient frontiers under the various measures, are the subset of boundary above the corresponding minimum global risk portfolios. We see that under $VaR_{5\%}$ and $ES_{5\%}$ the set of efficient portfolios is reduced with respect to the variance.

Mean-Value-at-Risk Efficient Frontier (alpha=5%)
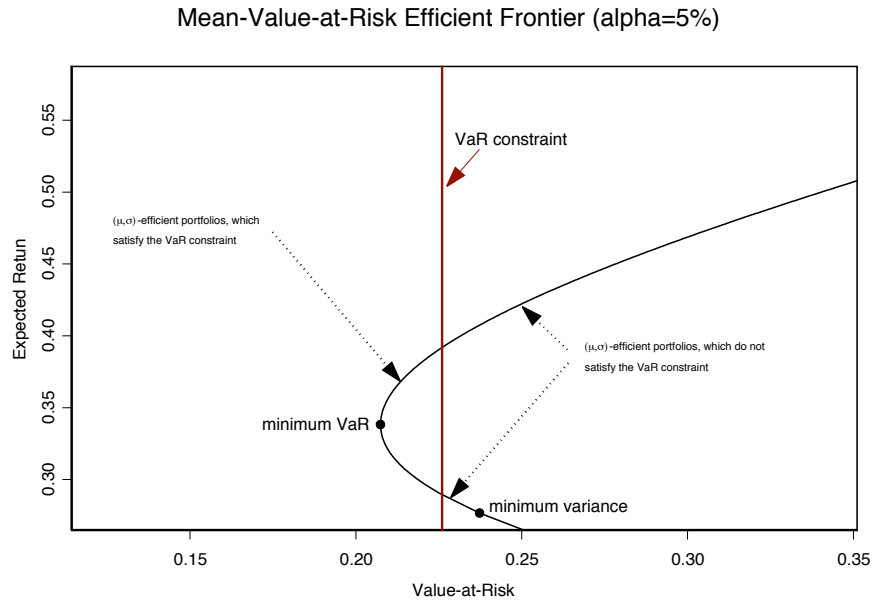
Figure 3: $(\mu, VaR_{5\%})$-boundary with the global minimum variance portfolio. Portfolios on the $(\mu, VaR_{5\%})$-boundary between the global minimum $VaR_{5\%}$ portfolio and the global minimum variance portfolio, are (μ, σ)-efficient. The VaR constraint (vertical line) could force (μ, σ) investors with high variance to reduce the variance and (μ, σ) investors with low variance to increase the variance, to be on the left side of the VaR constraint.

5 Portfolio optimization with risk-less asset and Tobin separation

In this section we take $\Delta = \Delta^K$ and we solve the portfolio optimization problem $(\mathcal{M}_{risk})$ introduced at the beginning of the previous section. As before we start our analysis with the problem $(\mathcal{M}_\sigma)$ and we extend some results to the case where the risk measure has the form $\gamma\sigma - \mu$ for some $\gamma > 0$. To make clear that we are now in an economy with the risk-less asset, we add to our previous notation for the $(\mu, risk)$-boundary, the $(\mu, risk)$-efficient frontier, ... an exponent R.

5.1 Mean-variance portfolio optimization with risk-free asset

The portfolio optimization $(\mathcal{M})_\sigma^R$ can be easily solved from the first order conditions. We make the same assumptions on V and $\boldsymbol{\mu}$ as in the previous section. We obtain that a portfolio $\boldsymbol{\lambda}^* \in \Delta^K$ is $(\mu, \sigma)^R$-optimal if and only if

$$\boldsymbol{\lambda}_1^* = \nu(\overline{\mu})\, V^{-1}(\boldsymbol{\mu_1} - \mu_0 \boldsymbol{e_1}), \tag{21}$$

$$\lambda^{*,0} = 1 - \boldsymbol{\lambda}_1' \boldsymbol{e_1}, \tag{22}$$

where $\nu(\overline{\mu}) = \frac{\overline{\mu} - \mu_0}{A - 2\mu_0 C + \mu_0^2 B}$. Let $D(\mu_0) = A - 2\mu_0 C + \mu_0^2 B$. We can easily show that $D(\mu_0) \geq D/B > 0$ and $D(\mu_0) = D/B$ if and only if $\mu_0 = C/B$. Note that $\nu(\mu_0) = 0$ and thus the optimal portfolio for μ_0 is the risk-free asset. The parameter $\overline{\mu}$ enters in equations (21) and (22) only through the function $\nu(\overline{\mu})$.
We can characterize the $(\mu, \sigma)^R$-boundary directly form equations (21) and (22) in the following way

$$(\sigma, \overline{\mu}) \in \mathcal{B}_{(\mu,\sigma)}^R \Longleftrightarrow \sigma = \frac{|\overline{\mu} - \mu_0|}{\sqrt{D(\mu_0)}} \Longleftrightarrow \overline{\mu} = \mu_0 \pm \sigma\sqrt{D(\mu_0)},\ \sigma \geq 0. \tag{23}$$

It is clear from the last equation that the $(\mu, \sigma)^R$-boundary is given by two straight lines in the $(\sigma, \overline{\mu})$-plane crossing in $(0, \mu_0)$. Moreover, from equation (23), we see that the lower branch of the $(\mu, \sigma)^R$-boundary is dominated by the upper branch, since optimal portfolios in the upper branch provide an higher return with the same risk. We have the following characterization of the $(\mu, \sigma)^R$-efficient frontier

$$(\sigma, \overline{\mu}) \in \mathcal{E}_{(\mu,\sigma)}^R \Longleftrightarrow \overline{\mu} = \mu_0 + \sigma\sqrt{D(\mu_0)},\ \sigma \geq 0. \tag{24}$$

We now address our attention to the following question: do can we find a portfolio in $\Delta^K \cap \Delta^{K-1}$ which is $(\mu, \sigma)^R$-efficient and simultaneously (μ, σ)-efficient? First, this portfolio should satisfy equations (11) and (21), i.e. it should be (μ, σ)-optimal and $(\mu, \sigma)^R$-optimal. Second it should have and expected return bigger than $\min\{\frac{C}{B}, \mu_0\}$ by Proposition 4.1 and equation (24). We start with equation (21), i.e. with an $(\mu, \sigma)^R$-optimal portfolio, and we impose that it belongs to Δ^{K-1}. Let denote this portfolio by $\boldsymbol{\lambda}_1^{\text{tang}}$. Since $\boldsymbol{\lambda}_1^{\text{tang}} \in \Delta^{K-1}$ we have

$$\boldsymbol{e_1}' \boldsymbol{\lambda}_1^{\text{tang}} = 1, \quad \text{and} \quad \lambda^{0,\text{tang}} = 0$$

and thus by equations (21) and (22) we find

$$\boldsymbol{\lambda}_1^{\text{tang}} = \frac{1}{C - \mu_0 B} V^{-1}(\boldsymbol{\mu_1} - \mu_0 \boldsymbol{e_1}). \tag{25}$$

The portfolio $\boldsymbol{\lambda}_1^{\text{tang}}$ is uniquely defined by this last equation and we can easily show that it also satisfies equation (11) with $\overline{\mu}^{\text{tang}} = \frac{A - \mu_0 C}{C - \mu_0 B}$, i.e. it is (μ, σ)-optimal. The last step of our

construction consists in verifying whether or not $\overline{\mu}^{\mathrm{tang}} > \min\{\frac{C}{B}, \mu_0\}$. If this last inequality holds, then $\boldsymbol{\lambda}_1^{\mathrm{tang}}$ is both (μ, σ)-efficient and $(\mu, \sigma)^R$-efficient. We have

Proposition 5.1. *The portfolio* $\boldsymbol{\lambda}_1^{\mathrm{tang}}$ *defined by equation (25) is* (μ, σ)-*efficient and* $(\mu, \sigma)^R$-*efficient if and only if* $\mu_0 < \frac{C}{B}$.

Proof. Follows directly from above, together with the last trivial step of verifying that $\overline{\mu}^{\mathrm{tang}} > \min\{\frac{C}{B}, \mu_0\}$ iff $\mu_0 < \frac{C}{B}$. □

The portfolio $\boldsymbol{\lambda}_1^{\mathrm{tang}}$ is called the *tangency or market portfolio*. The terminology *tangency portfolio* follows from the fact the $(\mu, \sigma)^R$-efficient frontier $\mathcal{E}_{(\mu,\sigma)}^R$ is tangent to the (μ, σ)-efficient frontier $\mathcal{E}_{(\mu,\sigma)}$ in the point $(\sigma^{\mathrm{tang}}, \overline{\mu}^{\mathrm{tang}})$, where σ^{tang} denotes the variance of the portfolio $\boldsymbol{\lambda}_1^{\mathrm{tang}}$. The pair $(\sigma^{\mathrm{tang}}, \overline{\mu}^{\mathrm{tang}})$ is the only intersection point of the two frontiers (see Figure 4).
From equation (25) we see that the tangency portfolio does not depend on the investors expected return $\overline{\mu}$, but only on assets' characteristics. Moreover, we can rewrite equation (21) and equation (22) as follows

$$\boldsymbol{\lambda}^* = (1 - \widetilde{\nu}(\overline{\mu})) \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} + \widetilde{\nu}(\overline{\mu}) \begin{pmatrix} 0 \\ \boldsymbol{\lambda}_1^{\mathrm{tang}} \end{pmatrix}. \tag{26}$$

where $\widetilde{\nu}(\overline{\mu}) = (C - \mu_0 B)\frac{\overline{\mu} - \mu_0}{D(\mu_0)}$. Note that $\widetilde{\nu}$ is an affine function of $\overline{\mu}$. We obtain the *Tobin separation*, which states that any $(\mu, \sigma)^R$-efficient portfolio is the combination of two other $(\mu, \sigma)^R$-efficient portfolios: the portfolio which invests only in the risk-free asset and the tangency portfolio. Therefore, homogeneous investors differ exclusively by the weights they put on these two portfolios, i.e. on the specific target expected return $\overline{\mu}$ they have. Under the assumption that market participants use mean-variance portfolio optimization with same beliefs about the probabilities (i.e. they are homogeneous), then in equilibrium the relative market capitalization of each risky asset is equal to the corresponding weight in the tangency portfolio (see Eichberger and Harper 1997). That's the reason why the tangency portfolio is often called market portfolio. Here, we would like to point out that Tobin separation holds independently from equilibrium consideration and that the tangency portfolio corresponds to the market portfolio only if all investors have the same beliefs about probabilities (and thus the same inputs for V and μ). From Tobin separation we also get that for all $(\mu, \sigma)^R$-efficient portfolio $\boldsymbol{\lambda}^* \in \Delta^K$,

$$\boldsymbol{\lambda}_1^{\mathrm{tang}} = \frac{1}{\boldsymbol{e}_1' \boldsymbol{\lambda}_1^*} \boldsymbol{\lambda}_1^*. \tag{27}$$

Proof. From equation (26) we obtain

$$\boldsymbol{\lambda}_1^* = \widetilde{\nu}(\overline{\mu}) \boldsymbol{\lambda}_1^{\mathrm{tang}} \quad \text{and} \quad \boldsymbol{e}_1' \boldsymbol{\lambda}_1^* = \widetilde{\nu}(\overline{\mu}) \underbrace{\boldsymbol{e}_1' \boldsymbol{\lambda}_1^{\mathrm{tang}}}_{=1} = \widetilde{\nu}(\overline{\mu}).$$

Equation (27) follows immediately. □

Tobin separation suggests an alternative (and desirable) way to formulate the mean-variance portfolio selection. We suppose that investors possess utility functions satisfying the following properties:

 (i) $U(R) = W(\sigma(R), \mu(R))$,

 (ii) W is strict quasi-concave,

 (iii) $\frac{\partial W}{\partial \sigma} < 0$ and $\frac{\partial W}{\partial \mu} > 0$.
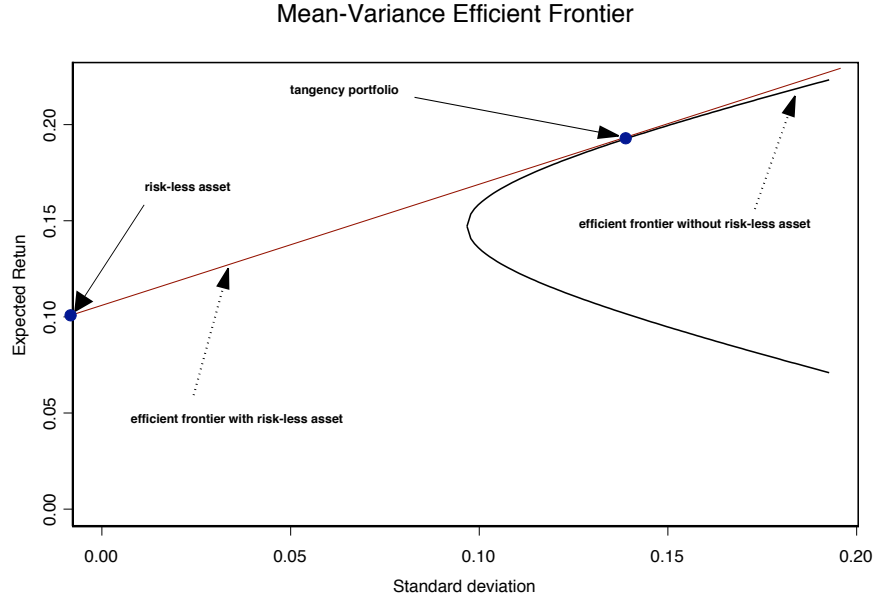
Figure 4: Mean-variance efficient frontier with and without risk-less asset. The mean-variance efficient frontier with risk-less asset is the linear combination of the risk-less asset and the tangency portfolio.

We denote the class of utility functions with (i)-(iii) by $\mathcal{U}^\sigma$. The first property just say that the utility function depends only on mean and variance. The second property ensures that indifference curves define strictly convex set in the (σ, μ)-plane and thus each investor prefers exactly one pair (σ, μ) on the efficient frontier. The third property is a sort of "rationality postulate" and asserts that investors utilities are strictly decreasing in σ and strictly increasing in μ, i.e. investors strictly prefer more return and less risk. The utility function $U^\xi(R) = W^\xi(\sigma(R), \mu(R)) = \mu(R) - \frac{\xi}{2}\sigma(R)^2$ for some parameter $\xi > 0$ satisfies the three properties (i)-(iii) and is quite simple for writing down the optimization problem.

Tobin separation ensures that investors, independently of the preferences they have within the class $\mathcal{U}^\sigma$, select a combination of risk-free asset and tangency portfolio. It follows that an alternative way for computing the tangency portfolio, consists in maximizing the utility of some investor satisfying (i)-(iii) over the space Δ^K and then, due to equation (27), we get the tangency portfolio by normalization. Since this results hold independently of the utility functions chosen in the class $\mathcal{U}^\sigma$, it makes sense to solve the maximization problem with a utility function in $\mathcal{U}^\sigma$ which can be easily handled analytically. We choose the function U^ξ for some positive ξ. We obtain the following optimization problem $(\mathcal{M}_\sigma^{W^\xi})$:

$$\max_{\boldsymbol{\lambda} \in \Delta^K} \boldsymbol{\lambda}'\boldsymbol{\mu} - \frac{\xi}{2}\boldsymbol{\lambda}_1'V\boldsymbol{\lambda}_1. \tag{28}$$

We can rewrite the objective function as

$$\boldsymbol{\lambda}'\boldsymbol{\mu} - \frac{\xi}{2}\boldsymbol{\lambda}_1'V\boldsymbol{\lambda}_1 = \mu_0 + (\boldsymbol{\mu}_1' - \mu_0\boldsymbol{e}_1')\boldsymbol{\lambda}_1 - \frac{\xi}{2}\boldsymbol{\lambda}_1'V\boldsymbol{\lambda}_1,$$

using that $\lambda^0 = 1 - \boldsymbol{\lambda'_1} e_1$. Therefore, the optimization problem $(\mathcal{M}_\sigma^{W^\xi})$ is equivalent to

$$\max_{\boldsymbol{\lambda_1} \in \mathbb{R}^K} \mu_0 + (\boldsymbol{\mu'_1} - \mu_0 \boldsymbol{e'_1}) \boldsymbol{\lambda_1} - \frac{\xi}{2} \boldsymbol{\lambda'_1} V \boldsymbol{\lambda_1}.$$

The first-order condition is given by

$$\mu_1 - \mu_0 \boldsymbol{e_1} = \xi V \boldsymbol{\lambda_1}$$

and thus

$$\boldsymbol{\lambda^*_1} = \xi^{-1} V^{-1} (\boldsymbol{\mu_1} - \mu_0 \boldsymbol{e_1}).$$

We normalize and we obtain the tangency portfolio of equation (25).

5.2 Mean-$risk^\gamma$ portfolio optimization with risk-free asset

We now consider the problem $(\mathcal{M}_{risk^\gamma}^R)$:

$$\min_{\boldsymbol{\lambda} \in \Delta^K} risk^\gamma(R_{\boldsymbol{\lambda_1}})$$
$$\text{s.t.} \quad \boldsymbol{\lambda'} \boldsymbol{\mu} = \overline{\mu},$$

where $risk^\gamma = \gamma\sigma - \mu$ for some $\gamma > 0$. Due to the particular form of $risk^\gamma$ we obtain the $(\mu, risk^\gamma)$-boundary

$$\mathcal{B}_{(risk^\gamma, \mu)}^R = \left\{ (risk^\gamma, \mu) \in \mathbb{R}^2 \Big| \left(\frac{risk^\gamma + \mu}{\gamma}, \mu \right) \in \mathcal{B}_{(\mu, \sigma)}^R \right\},$$

i.e.

$$(risk^\gamma, \overline{\mu}) \in \mathcal{B}_{(\mu, risk^\gamma)}^R \Longleftrightarrow risk^\gamma = \begin{cases} \frac{\overline{\mu}(\gamma - \sqrt{D(\mu_0)}) - \gamma\mu_0}{\sqrt{D(\mu_0)}} & \text{if} \quad \overline{\mu} \geq \mu_0, \\ \frac{\overline{\mu}(-\gamma - \sqrt{D(\mu_0)}) + \gamma\mu_0}{\sqrt{D(\mu_0)}} & \text{if} \quad \overline{\mu} < \mu_0. \end{cases} \tag{29}$$

It follows that geometrically the $(\mu, risk^\gamma)^R$-efficient frontier is given by two straight lines in the $(risk^\gamma, \overline{\mu})$-plane, with intersection $(-\mu_0, \mu_0)$.

We resume in the following Proposition the main results for the $(\mu, risk^\gamma)^R$ portfolio optimization.

Proposition 5.2. *(i) If $\gamma > \sqrt{D(\mu_0)}$ then a portfolio is $(\mu, risk^\gamma)^R$-efficient if and only if it is $(\mu, \sigma)^R$-efficient.*

(ii) If $\gamma \leq \sqrt{D(\mu_0)}$ then no $(\mu, risk^\gamma)$-efficient portfolio exists.

Proof. For $\gamma > \sqrt{D(\mu_0)}$, by equation (29), the lower branch of the $(\mu, risk^\gamma)$-boundary is dominated by the upper branch. Thus the $(\mu, risk^\gamma)^R$-efficient frontier is given by the upper branch of the boundary. Each $(\mu, \sigma)^R$-efficient portfolio generates a pair $(\mu, risk^\gamma)$ which belongs to the $(\mu, risk^\gamma)^R$-efficient frontier. Moreover, a pair $(\mu, risk^\gamma)$ in the $(\mu, risk^\gamma)^R$-efficient frontier belongs to a $(\mu, risk^\gamma)^R$-efficient portfolio, which is also $(\mu, \sigma)^R$-efficient.

For $\gamma \leq \sqrt{D(\mu_0)}$, the $(\mu, risk^\gamma)$-boundary degenerates (both the upper and the lower branch are geometrically given by a straight line with negative slope), and one could still decrease the risk by simultaneously increasing the return, thus no $(\mu, risk^\gamma)$-efficient portfolio can exist. $\square$

Corollary 5.1. *If $\gamma > \sqrt{D(\mu_0)}$ and $\mu_0 < \frac{C}{B}$, then the tangency portfolio $\boldsymbol{\lambda}_1^{\mathrm{tang}}$ defined by equation (25) is the unique portfolio which is $(\mu, risk^\gamma)$-efficient and $(\mu, risk^\gamma)^R$-efficient.*

Proof. If $\gamma > \sqrt{D(\mu_0)}$, then $\gamma > \sqrt{\frac{D}{B}}$. Thus by Proposition 4.3 the set of $(\mu, risk^\gamma)$-efficient portfolios is the the subset of (μ, σ)-efficient portfolios with expected return greater or equal $\frac{C}{B} + \sqrt{\frac{D}{B}\left(\frac{\gamma^2}{B\gamma^2 - D} - \frac{1}{B}\right)}$. The expected return of the tangency portfolio corresponds to $\frac{A - \mu_0}{C - \mu_0}$ and it can be easily shown that this value is larger than $\frac{C}{B} + \sqrt{\frac{D}{B}\left(\frac{\gamma^2}{B\gamma^2 - D} - \frac{1}{B}\right)}$ if $\mu_0 < \frac{C}{B}$ and $\gamma > \sqrt{D(\mu_0)}$. Thus the tangency portfolio is $(\mu, risk^\gamma)$-efficient. Moreover, by the previous Proposition, if $\gamma > \sqrt{D(\mu_0)}$ then each $(\mu, \sigma)^R$-efficient portfolio is also $(\mu, risk^\gamma)^R$-efficient.
Suppose now that another portfolio is $(\mu, risk^\gamma)$-efficient and $(\mu, risk^\gamma)^R$-efficient. Then by the previous Proposition and by Proposition 4.3 it must also be (μ, σ)-efficient and $(\mu, \sigma)^R$-efficient. But this contradicts the fact that the tangency portfolio is the unique (μ, σ)-efficient and $(\mu, \sigma)^R$-efficient portfolio. $\qquad\square$

The Corollary implies that Tobin separation also holds for $(\mu, risk^\gamma)$ investors, i.e. for γ big enough the efficient asset allocation of a $(\mu, risk^\gamma)$ investors is given by a combination of the risk-free asset and the tangency portfolio. The same statement can be also obtained by considering the utility function approach already introduced for the mean-variance case. Let us consider a utility function $\widetilde{U}$ with the following properties:

(i) $\widetilde{U}(R) = \widetilde{W}(risk^\gamma(R), \mu(R))$,

(ii) $\widetilde{W}$ is strict quasi-concave,

(iii) $\frac{\partial \widetilde{W}}{\partial risk^\gamma} < 0$ and $\frac{\partial \widetilde{W}}{\partial \mu} > 0$.

We denote the set of utility function with (i)-(iii) by $\mathcal{U}^{risk^\gamma}$. We define the function $W(\sigma, \mu) = \widetilde{W}(risk^\gamma, \mu) = \widetilde{W}(\gamma\sigma - \mu, \mu)$. Then U given by $U(R) = W(\sigma(R), \mu(R))$ belongs to $\mathcal{U}^\sigma$ and

(i) $\frac{\partial W}{\partial \sigma} = \gamma \frac{\partial \widetilde{W}}{\partial risk^\gamma}$,

(ii) $\frac{\partial W}{\partial \mu} = \frac{\partial \widetilde{W}}{\partial \mu} - \frac{\partial \widetilde{W}}{\partial risk^\gamma}$.

Let $\boldsymbol{\lambda} = (\lambda^0, \boldsymbol{\lambda}_1') \in \Delta^K$, then

$$
\begin{aligned}
\mu_{\boldsymbol{\lambda}} &= \boldsymbol{\mu}'\boldsymbol{\lambda}, \\
risk_{\boldsymbol{\lambda}}^\gamma &= risk^\gamma(R_{\boldsymbol{\lambda}}) = \gamma\left(\boldsymbol{\lambda}_1' V \boldsymbol{\lambda}_1\right)^{\frac{1}{2}} - \boldsymbol{\mu}'\boldsymbol{\lambda}.
\end{aligned}
$$

The optimization problem for the $(\mu, risk^\gamma)$ investor with utility function $\widetilde{W}$ is $(\mathcal{M}_{risk^\gamma}^{\widetilde{W}})$:

$$
\max_{\boldsymbol{\lambda} \in \Delta^K} \widetilde{W}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}}). \tag{30}
$$

The first-oder-conditions are:

$$
\frac{\partial \widetilde{W}}{\partial risk^\gamma}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}})\frac{\partial}{\partial \lambda^j}risk_{\boldsymbol{\lambda}}^\gamma + \frac{\partial \widetilde{W}}{\partial \mu}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}})\frac{\partial}{\partial \lambda^j}\mu_{\boldsymbol{\lambda}} + \theta = 0, \quad \text{for } j = 0, 1, \dots, K,
$$

where θ is the Lagrange multiplier. These are equivalent to:

$$\frac{\partial \widetilde{W}}{\partial risk^\gamma}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}}) \left(\gamma(\boldsymbol{\lambda_1'} V \boldsymbol{\lambda_1})^{-\frac{1}{2}} (V \boldsymbol{\lambda_1})_j - \mu^j \right) + \frac{\partial \widetilde{W}}{\partial \mu}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}}) \mu^j + \theta = 0, \quad j = 1, \ldots, K,$$

$$\frac{\partial \widetilde{W}}{\partial risk^\gamma}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}})(-\mu_0) + \frac{\partial \widetilde{W}}{\partial \mu}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}})\mu_0 + \theta = 0.$$

We rewrite the left hand-side of the first K equations using the vector notation. From the second equation we obtain θ; it follows:

$$
\begin{aligned}
0 &= \frac{\partial \widetilde{W}}{\partial risk^\gamma}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}}) \left(\gamma(\boldsymbol{\lambda_1'} V \boldsymbol{\lambda_1})^{-\frac{1}{2}} V \boldsymbol{\lambda_1} - \boldsymbol{\mu_1} \right) + \frac{\partial \widetilde{W}}{\partial \mu}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}}) \boldsymbol{\mu_1} + \\
&\quad + \left(\frac{\partial \widetilde{W}}{\partial risk^\gamma}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}}) - \frac{\partial \widetilde{W}}{\partial \mu}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}}) \right) \mu_0 \boldsymbol{e_1} = \\
&= \left(-\frac{\partial \widetilde{W}}{\partial risk^\gamma}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}}) + \frac{\partial \widetilde{W}}{\partial \mu}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}}) \right) (\boldsymbol{\mu_1} - \mu_0 \boldsymbol{e_1}) + \\
&\quad + \frac{\partial \widetilde{W}}{\partial risk^\gamma}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}}) \gamma(\boldsymbol{\lambda_1'} V \boldsymbol{\lambda_1})^{-\frac{1}{2}} V \boldsymbol{\lambda_1} = \\
&= \frac{\partial W}{\partial \mu}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}})(\boldsymbol{\mu_1} - \mu_0 \boldsymbol{e_1}) + \frac{\partial W}{\partial \sigma}(risk_{\boldsymbol{\lambda}}^\gamma, \mu_{\boldsymbol{\lambda}})(\boldsymbol{\lambda_1'} V \boldsymbol{\lambda_1})^{-\frac{1}{2}} V \boldsymbol{\lambda_1} = \\
&= \frac{\partial W}{\partial \mu}(\sigma_{\boldsymbol{\lambda}}, \mu_{\boldsymbol{\lambda}}) \nabla_{\boldsymbol{\lambda_1}} \mu_{\boldsymbol{\lambda}} + \frac{\partial W}{\partial \sigma}(\sigma_{\boldsymbol{\lambda}}, \mu_{\boldsymbol{\lambda}}) \nabla_{\boldsymbol{\lambda_1}} \sigma_{\boldsymbol{\lambda}} - \frac{\partial W}{\partial \mu}(\sigma_{\boldsymbol{\lambda}}, \mu_{\boldsymbol{\lambda}}) \mu_0 \boldsymbol{e_1}.
\end{aligned}
$$

This last equation together with

$$\frac{\partial W}{\partial \mu}(\sigma_{\boldsymbol{\lambda}}, \mu_{\boldsymbol{\lambda}}) \frac{\partial}{\partial \mu_0} \mu_{\boldsymbol{w}} + \theta = 0,$$

are exactly the first-order-conditions for the problem $(\mathcal{M}_\sigma^W)$. We know from the previous section that the normalized solution of $(\mathcal{M}_\sigma^W)$ is the tangency portfolio.

We come back to our original question about the efficient frontiers of a (μ, VaR_α) and a (μ, ES_α) investor in an economy with a risk-free asset. In this subsection we have shown that if the returns are multivariate Gaussian distributed, then some of the results we obtain in a (μ, σ) "world" remain unchanged when we use VaR_α or ES_α. This follows from the fact that both VaR_α and ES_α have the form $risk^\gamma$ for some γ. The exact expression for the parameter γ has been given in the previous section. More precisely, the sets of efficient portfolios in the presence of a risk-free asset remain unchanged under the various risk measures considered in this work, on condition that all efficient sets are not empty (see Proposition 5.2).

Since long horizon returns are often well fitted by the normal distribution, we suggests that $(\mu, risk)$ portfolio selection under VaR_α or ES_α does not really represent an improvement with respect to the classical (μ, σ) approach. Naturally this is true only when the portfolio returns are multivariate Gaussian distributed or can be well approximated by a Gaussian distribution. In the next section we introduce a general framework for portfolio decision with ES_α. We will present an example based on a daily dataset from the Swiss Market Index and we will see that the (μ, ES_α) portfolio decision look very similar to the (μ, σ) asset allocation.

6 Generalization

Up to now we have assumed that returns are normally distributed. This assumption simplify the analytical formulation of the mean-$risk$ optimization problem, where $risk$ can be taken to be value-at-risk, expected-shorfall or every measure of the form $risk^\gamma(R_{\boldsymbol{\lambda}}) = \gamma\sigma_{\boldsymbol{\lambda}} - \mu_{\boldsymbol{\lambda}}$. In the general case where return are not normally distributed, the mean-$risk$ problem maybe provides degenerated solutions, as shown by Lemus Rodriguez (1999).

We consider the mean-ES_α optimization problem under the assumption that returns have continuous distribution functions. In this case ES_α coincides with TCE_α and by Lemma 3.2 we have

$$ES_\alpha(R_{\boldsymbol{\lambda}}) = \inf\{a + \frac{1}{\alpha}\mathbb{E}\big[(-R_{\boldsymbol{\lambda}} - a)^+\big] \,|\, a \in \mathbb{R}\}.$$

Let $F_\alpha(R_{\boldsymbol{\lambda}}, a) = a + \frac{1}{\alpha}\mathbb{E}\big[(-R_{\boldsymbol{\lambda}} - a)^+\big]$. Rockafellar and Uryasev (1999) show that

$$\min_{\boldsymbol{w}\in\Delta} ES_\alpha(\mathbf{R}_{\boldsymbol{w}}) = \min_{(\boldsymbol{w},a)\in(\Delta\times\mathbb{R})} F_\alpha(\mathbb{R}_{\boldsymbol{w}}, a), \tag{31}$$

where $\Delta \subset \mathbb{R}^K$ denotes as usual the convex set of available portfolios. The optimization problem $(\mathcal{M}_{ES_\alpha})$ can be written as follows:

$$\min_{(\boldsymbol{\lambda_1},a)\in\Delta\times\mathbb{R}} a + \frac{1}{\alpha}\mathbb{E}\big[(-R_{\boldsymbol{\lambda_1}} - a)^+\big]$$
$$\text{s.t.} \quad \mathbb{E}\big[R_{\boldsymbol{\lambda_1}}\big] \geq \overline{\mu}.$$

We introduce an auxiliary random variable $Z \geq 0$. Then an equivalent formulation of the previous problem is the following $(\mathcal{M}^*_{ES_\alpha})$

$$\min_{(\boldsymbol{\lambda},a,Z)\in\Delta\times\mathbb{R}\times L^0(\mathbb{R})} \quad a + \frac{1}{\alpha}\mathbb{E}\big[Z\big]$$
$$\text{s.t.} \quad \mathbb{E}\big[R_{\boldsymbol{\lambda}}\big] \geq \overline{\mu}$$
$$Z \geq -R_{\boldsymbol{\lambda}} - a, \quad \text{with probability 1}$$
$$Z \geq 0.$$

Proof. Note that $Z = (-R_{\boldsymbol{\lambda}} - a)^+$ satisfies the constraints of $(\mathcal{M}^*_{ES_\alpha})$. Moreover, whenever a triple $(\boldsymbol{\lambda},a,Z) \in (\Delta \times \mathbb{R} \times L^0(\mathbb{R}))$ satisfies the constraints of $(\mathcal{M}^*_{ES_\alpha})$, then

$$a + \frac{1}{\alpha}\mathbb{E}\big[Z\big] \geq a + \frac{1}{\alpha}\mathbb{E}\big[(-R_{\boldsymbol{\lambda}} - a)^+\big]$$

and the inequality is strict if $\mathbb{P}\big[Z > (-R_{\boldsymbol{\lambda}} - a)^+\big] > 0$. Thus the minimum in $(\mathcal{M}^*_{ES_\alpha})$ is attained for some triple $(\boldsymbol{\lambda}^*, a^*, Z^*)$ where

$$\mathbb{P}\big[Z^* = (-R_{\boldsymbol{\lambda}^*} - a^*)^+\big] = 1.$$

This complete the proof. □

The advantage of this formulation is that we obtain a linear program, with convex contraints. This assure that independently of the distribution of $R_{\boldsymbol{\lambda}}$, the solution set is a convex poyhedron. This makes the mean-ES_α criterion more attractive than the mean-VaR_α criterion. The latter in fact can only be reduced to a non-convex problem and thus several local minima may occur. The disadvantage of the mean-ES_α problem formulated above is that it has usually infinite dimension,

since whenever the sample space is not finite, the set $\{\omega \in \Omega \mid Z(\omega) \geq -R_{\boldsymbol{\lambda}}(\omega) - a\}$ is also not finite. For practical purposes, we consider a discrete version of $(\mathcal{M}^*_{ES_\alpha})$. Let $\mathbf{R}$ be the vector of asset returns and $\boldsymbol{\xi}_i \in \mathbb{R}^K$ (or $\mathbb{R}^{K+1}$ if the risk-free asset exists), for $i = 1, \cdots, N$ be N realizations of $\mathbf{R}$. Moreover, we assume that each $\boldsymbol{\xi}_i$ has probability $\frac{1}{N}$ to occur.

The discrete version of the optimization problem is the following

$$\min_{(\boldsymbol{\lambda}, a, \mathbf{z}) \in \Delta \times \mathbb{R} \times \mathbb{R}^N} \quad a + \frac{1}{\alpha N} \sum_{i=1}^{N} z_i$$

$$\text{s.t.} \quad \boldsymbol{\lambda}' \boldsymbol{\mu} \geq \overline{\mu}$$

$$z_i \geq -\boldsymbol{\lambda}' \boldsymbol{\xi}_i - a, \quad \text{for } i = 1, \ldots, N$$

$$z_i \geq 0, \quad \text{for } i = 1, \ldots, N.$$

This last problem is a finite dimensional linear optimization problem and can be solved using a linear program algorithm. Note that one can also include short-sell constraints or lower-upper bounds for the weights λ^i without changing the structure of the problem.

We apply this procedure to our data set, consisting in the 252 daily observations in 1999 of 19 stocks in the Swiss Market Index and the risk-free asset. We compute the optimal strategies for different values of $\overline{\mu}$ and a given fixed α. Table 1 shows the results for some level of $\overline{\mu}$ and Figure 5 gives the efficient frontier with and without risk-free asset. The tangency portfolio from the (μ, σ)-analysis corresponds to the tangency portfolio in the $(\mu, ES_{1\%})$-efficient frontier, as one would expect in the multivariate Gaussian case.

$\overline{\mu}$	0.2	0.1	0.05
ES_α	2.772	1.261	0.506
Risk-free	28.8	67.4	86.7
ABB Ldt	26.2	12.0	4.9
Adecco	4.7	2.1	0.9
Holderbank	12.8	5.9	2.4
Swatch	27.5	12.6	5.1

Table 1: Optimal strategy in percentage with ES_α, $\alpha = 1\%$. We impose short-sale constraints. All other 15 stock has not been selected.

7 Conclusion

In this work we have analyzed the portfolio selection problem following the mean-*risk* approach, where the *risk* measure takes the three forms standard deviation, value-at-risk and expected shortfall. We have shown that under the assumption of multivariate Gaussian distributed returns, the set of efficient portfolios under value-at-risk and expected shortfall is a subset of the set of efficient portfolios under the standard deviation: (μ, σ)-portfolio selection could be inefficient under value-at-risk or expected shortfall, but the opposite never occurs. Moreover, the set of efficient portfolios under value-at-risk is a proper subset of the set of efficient frontier under expected shortfall. We have also shown that (μ, VaR_α) and (μ, ES_α) efficient frontiers could be empty for value of α greater than a given level. This suggests that the choice of the level α needs some precaution. In the presence of a risk-free asset, the set of efficient portfolios under the various risk measures are identical, unless one of these is empty. This allows an extension of Tobin separation in the case of (μ, VaR_α) or (μ, ES_α) portfolio selection. Finally, using a general procedure for
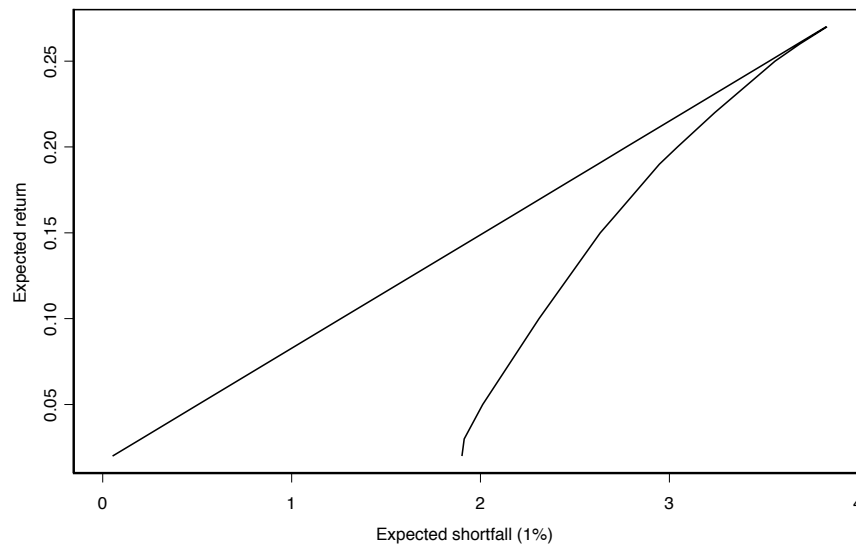
Figure 5: Mean-ES_α efficient frontier with and without risk-less asset.

portfolio selection under expected-shortfall, we have computed the portfolio optimization for a date set from the Swiss Market Index and we obtained an asset allocation similar to the mean-variance allocation.

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