An Overlapping Generations Model for Monetary Policy Analysis

Samuel Huber and Jaehong Kim

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Abstract

We integrate an overlapping generations model into a new monetarist framework and show that the Friedman rule is not optimal. This is because inflation makes saving for retirement less attractive, such that young agents optimally choose to increase their consumption at the expense of lower savings. On the other hand, old agents consume less due to the inflation tax. We show that for low inflation rates, the former effect dominates the latter, such that the Friedman rule is not optimal. However, this effect disappears for higher inflation rates such that the optimal rate is at an intermediate level.

1 Introduction

In order to improve welfare, central banks seek to choose the optimal short-term nominal interest rate. In practice, this optimal interest rate is believed to be at an intermediate positive level, while theory has suggested for a long time that the Friedman (1969) rule is optimal, i.e., where nominal interest rates equal zero. Yet, theory has developed since Milton Friedman's seminal contribution, and the incorporation of additional frictions has allowed to justify a positive optimal nominal interest rate. We contribute to this theoretical literature by showing that only one additional assumption is necessary to achieve a welfare-improving role of inflation in the well-known Lagos and Wright (2005) framework and that the optimal inflation rate is at an intermediate level.[1] Concretely, we integrate an overlapping generations structure and assume

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[1]In our model, the nominal interest rate satisfies the Fisher equation. Therefore, it is equivalent for the central bank to set the nominal interest rate or the inflation rate.

that old/retired agents suffer a higher disutility of labor than young agents. As natural as this assumption seems, it turns out to have non-trivial implications concerning monetary policy.

In our model, agents can choose to save money for their retirement in order to benefit from their lower disutility when young. The lower the opportunity cost of holding money is (i.e., the lower nominal interest rates are), the less costly it is to save money. Therefore, agents reduce their consumption when young in order to increase their precautionary savings for retirement. Increasing the opportunity cost of holding money induces agents to reduce their savings; i.e., to increase their consumption when young. In contrast to this, old agents consume less when inflation increases, since they bear the inflation tax. We show that for low nominal interest rates, the former effect dominates the latter and thereby produces a welfare-enhancing role of inflation. However, we also show that this effect disappears for higher inflation rates, such that both young and old agents consume less as inflation increases further. Therefore, the optimal inflation rate is at an intermediate level.

We are not the first to show that it can be welfare-improving to increase inflation above the Friedman rule. For instance, the monetary models of Bewley (1980, 1983) show that inflation can be welfare-improving due to its distributional effects.[2] While some search models of money also feature distributional effects, this welfare-enhancing role of inflation is generally absent in models that build on Lagos and Wright (2005) due to the assumption of quasi-linear preferences.[3] However, for instance, by endogenizing participation decisions, several studies find a positive relationship between inflation and welfare in the Lagos and Wright (2005) framework.[4] There are also several studies that use overlapping generations models, which find that certain frictions produce a welfare-improving role of inflation. Beginning with the seminal contribution by Grandmont (1985) which assumes differences in risk aversion, more recent studies assume financial intermediation and limited communication, or type-dependent lump-sum transfers of money.[5]

In standard overlapping generations models, increasing inflation hurts old agents, but benefits young agents (e.g., Grandmont, 1985). However, we find that this result does not hold anymore in our model when the inflation rate exceeds a certain threshold. Hence, when inflation is sufficiently high, increasing inflation lowers consumption for both young and old agents (e.g., Lagos and Wright, 2005). It should therefore be clear that our mechanism is very different from previous studies and has nothing to do with the distributional effect of inflation.

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[2]Also see Levine (1991) and Kehoe et al. (1992).

[3]See, for instance, Chiu and Molico (2014); Deviatov and Wallace (2001); Menzio et al. (2013); Molico (2006) and Rocheteau et al. (2015a) about the distributional effects of inflation in search models.

[4]Here, we refer to Rocheteau and Wright (2005, 2009) and Rocheteau and Rodriguez-Lopez (2014). See also Cooley and Quadrini (2004) for a framework where money is introduced via cash-in-advance and Berentsen et al. (2007), who endogenize participation decisions in a model which builds on Shi (1997). Others integrate more rounds of trade in the Kiyotaki-Wright market, relax the assumption of quasi linear preferences, integrate limited access to credit, or introduce additional frictions to the Arrow-Debreu market, such as limiting agents' endowment of labor or limiting their participation to this market. See, for instance, Berentsen et al. (2005); Chiu and Molico (2010, 2011); Gomis-Porqueras and Sanches (2013); Rocheteau et al. (2015b) and Williamson (2006).

[5]See, for instance, Bhattacharya et al. (2005); Schreft and Smith (1997, 2002); Smith (2002) and Zhu (2008). We present more details of Grandmont's framework and compare it to our model in the discussion section.

2   Environment

Time is discrete and the economy is populated by overlapping generations who live for two periods. In each period, there are two markets that open sequentially. The first market is a centralized market, where agents produce and consume a general good and rebalance their money holdings. The second market is a goods market, where agents produce or consume a specialized good. Both goods are perishable, which rules out any form of commodity money.

At the beginning of each period, a generation g_t of young agents is born, where the time-subscript t denotes the period of birth. Young agents are born without any endowment. The measure of old agents equals g_{t-1}.

The centralized market is frictionless and agents consume and produce the general good. They enjoy utility $U(x)$ from x consumption, where $U'(x)$, $-U''(x) > 0$, $U'(0) = \infty$, and $U'(\infty) = 0$. The linear production technology transforms ℓ units of labor into ℓ units of the general good, which produces a disutility of labor. We assume that the disutility of labor differs between age and that old agents suffer a higher disutility than young agents. Equivalently, this can be interpreted as old agents having a higher utility of leisure. We assume that the disutility of old agents equals $\mu\ell$, where $\mu > 1$. The disutility of young agents is given by $f(\ell)$, where f is in $C^2(\mathbb{R})$, $f(0) = 0$, $f''(\ell) \geq 0$, and

$$\begin{cases} f'(\ell) = 1, & \text{for } \ell \leq \mathfrak{L} - \varepsilon, \\ f'(\ell) \in (1, \mu), & \text{for } \mathfrak{L} - \varepsilon < \ell < \mathfrak{L} + \varepsilon \\ f'(\ell) = \mu, & \text{for } \ell \geq \mathfrak{L} + \varepsilon, \end{cases}$$

for some $\mathfrak{L} > 0$ and small $\varepsilon > 0$. Figure 1 stylistically shows the evolution of the disutility of labor of young and old agents as a function of units of labor ℓ.
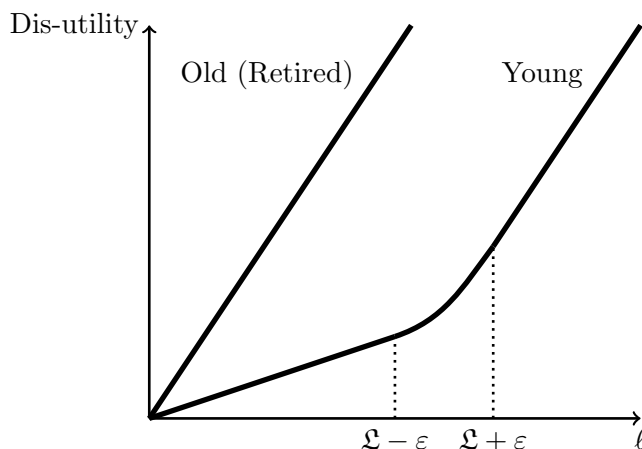

FIGURE 1: DISUTILITY OF LABOR

In the centralized market, a central bank supplies a perfectly storable, divisible, intrinsically

useless object, called money. The gross growth rate of money is given by $\gamma_t = M_{t+1}/M_t$, where M_t (M_{t+1}) denotes the stock of money in period t ($t+1$). Money is injected (withdrawn) through a lump-sum transfer T_t to all agents, where $T_t = M_t - M_{t-1} = (\gamma_{t-1} - 1)M_{t-1}$. Hereafter, we often use $i_t = (\gamma_t - \beta)/\beta$ to represent monetary policy, where i_t denotes the nominal interest rate on a risk-free one-period bond issued in the centralized market. It is evident that the risk-free nominal interest rate satisfies the Fisher equation, and that the central bank controls i_t by choosing γ_t.[6]

Before the goods market opens, young agents receive an idiosyncratic i.i.d. preference shock that determines their ability to produce or consume the specialized good in the goods market: With probability n, a young agent can consume but not produce, and with probability $1 - n$ he can produce but not consume. For simplicity of the settings, we assume that old agents can consume but not produce in the goods market.[7] We refer to consumers as buyers and to producers as sellers. The goods market is competitive and agents enjoy utility $u(q)$ from q consumption, where $u'(q), -u''(q) > 0$, $u'(0) = \infty$, $u'(\infty) = 0$, and $\alpha(q) := -qu''(q)/u'(q) \leq 1$ denotes the coefficient of risk aversion of $u(q)$. Sellers incur a utility cost $c(q) = q$ from producing q units of the specialized good. Due to anonymity and the absence of a record-keeping technology, sellers ask for immediate compensation; i.e., a medium of exchange is required and only money can serve this role. Young agents discount between two consecutive periods at the rate $\beta \in (0,1)$. For ease of understanding, Figure 2 shows the timing of events for a

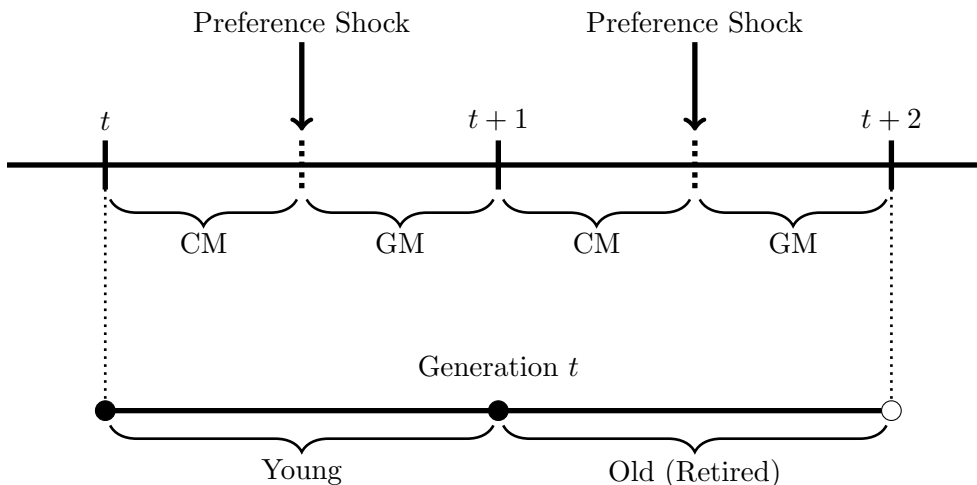FIGURE 2: TIME LINE

representative generation t, where CM denotes centralized market and GM denotes for goods market.

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[6]We do not incorporate the trading of such a risk-free bond in our model.

[7]Otherwise, some of the old agents would die with money, such that we would need additional settings concerning the lump-sum money transfer (bequests), which do not change our results qualitatively.

2.1 Planner Allocation

For a benchmark, it is helpful to derive the planner allocation. The social planner can dictate consumption and production and treats all agents symmetrically. He solves the following decision problem:

$$\max_{x_y, x_o, q_y, q_o} g[U(x_y) - f[(gx_y + g_{-1}x_o)/g] + nu(q_y) - (1-n)q^s] + g_{-1}[U(x_o) + u(q_o)],$$

subject to the market clearing constraint, $ngq_y + g_{-1}q_o = (1-n)gq^s$. The planner takes technology and preferences as given, and since young agents suffer a lower disutility of labor than old agents, we impose the assumption, without loss of generality, that only young agents work. The efficient allocation satisfies $u'(q_y) = u'(q_o) = 1$, and $U'(x_o) = U'(x_y) = f'[(gx_y + g_{-1}x_o)/g] \geq 1$.

3 The Decisions of Generation t

Hereafter, we focus on the decision problem of a representative generation t. We first present the last market (goods market) for an old agent in period $t+1$ and then work backwards to the first market (centralized market) for a young agent in period t. To economize on notation, we omit the time subscript t going forward. Next-period variables are indexed by $+1$, and previous-period variables by -1.

3.1 Period $t+1$: Old Agents

Goods Market In the goods market, all agents take prices as given. In period $t+1$, every old agent in the goods market buys the specialized good $q_{o,+1}$ at price p_{+1} and enjoys utility $u(q_{o,+1})$ from $q_{o,+1}$ consumption. Since an old agent dies at the end of period $t+1$, he will always spend all his money on consumption in this market. Therefore, when he enters the goods market with $m_{o,t+1}$ units of money, his value function equals

$$W_o(\phi_{+1}m_{o,+1}) := u(q_{o,+1}), \qquad \text{with} \qquad m_{o,+1} = p_{+1}q_{o,+1},$$

where ϕ_{+1} is the price of money in terms of centralized market goods in period $t+1$.

Centralized Market In the centralized market, agents can produce and consume the general good $x_{o,+1}$, work $\ell_{o,+1}$, receive the lump-sum money transfer T_{+1}, and save $m_{o,+1}$ for consumption in the goods market. An old agent entering the centralized market with $\hat{m}_{o,+1}$ units of money has the value function $V_o(\phi_{+1}\hat{m}_{o,+1})$ and solves the following decision problem in period $t+1$:

$$V_o(\phi_{+1}\hat{m}_{o,+1}) := \max_{x_{o,+1}, \ell_{o,+1}, m_{o,+1}} \{U(x_{o,+1}) - \mu\ell_{o,+1} + W_o(\phi_{+1}m_{o,+1})\},$$

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subject to
$$x_{o,+1} + \phi_{+1}m_{o,+1} = \ell_{o,+1} + \phi_{+1}\hat{m}_{o,+1} + \phi_{+1}T_{+1}.$$

Due to the quasi-linear preferences, the choice of $m_{o,+1}$ is independent of $\hat{m}_{o,+1}$ and thus each old agent exits the centralized market with the same amount of money.

3.2 Period t: Young Agents

Goods Market In the goods market, a young agent who is a seller with m_y units of money has the following decision problem:

$$W_y^s(\phi m_y) := \max_{q^s}\{-q^s + \beta V_o(\phi_{+1}\hat{m}_{o,+1})\}$$

subject to
$$\hat{m}_{o,+1} = m_y + pq^s,$$

where $W_y^s(\phi m_y)$ denotes his value function. Since the goods market is competitive, the price adjusts to equate aggregate demand and aggregate supply. The market clearing condition equals

$$ngq_y + g_{-1}q_o = (1-n)gq^s,$$

where ngq_y $(g_{-1}q_o)$ is the consumed quantity of a young (old) buyer times the number of buyers in his generation, and $(1-n)gq^s$ is the produced quantity of a young seller times the number of sellers in his generation.

A young agent who is a buyer with m_y units of money has the following decision problem:

$$W_y^b(\phi m_y) := \max_{q_y}\{u(q_y) + \beta V_o(\phi_{+1}\hat{m}_{o,+1})\}$$

subject to
$$pq_y \le m_y, \qquad \text{and} \qquad \hat{m}_{o,+1} = m_y - pq_y,$$

where $W_y^b(\phi m_y)$ denotes his value function. The first constraint states that a young buyer is constrained by his money holdings, while the second constraint states that the money he brings into the next period, $\hat{m}_{o,+1}$, equals the money he brought into the goods market less the amount he spends on consumption, $m_y - pq_y$.

Let $W_y(\phi m_y)$ denote the value function of a young agent with m_y units of money before the realization of the idiosyncratic preference shocks. Then, we have

$$W_y(\phi m_y) := nW_y^b(\phi m_y) + (1-n)W_y^s(\phi m_y).$$

With probability n, an agent becomes a buyer and his value from entering the goods market equals $W_y^b(\phi m_y)$. With probability $1-n$, he becomes a seller and his value function equals $W_y^s(\phi m_y)$.

Centralized Market A young agent entering the centralized market has no endowments, so he faces the following decision problem in period t:

$$V_y := \max_{x_y, \ell_y, m_y} \{U(x_y) - f(\ell_y) + W_y(\phi m_y)\},$$

subject to

$$x_y + \phi m_y = \ell_y + \phi T,$$

where young agents consume the general good x_y, work ℓ_y, receive the lump-sum transfer T, and save money m_y for the future.

4 Equilibrium

We focus on symmetric stationary monetary equilibria, where all agents follow identical strategies and real variables are constant over time. We also impose an additional assumption that the population is constant over time; i.e., $g_t = 1$ for all t. In stationary equilibrium, the real stock of money is constant, so that $\phi_t M_t = \phi_{t+1} M_{t+1}$, which implies

$$\beta(1 + i) = \gamma = \phi/\phi_{+1}. \tag{1}$$

In any monetary equilibrium, we obtain the following two equilibrium conditions from the old agent's decision problem (see Appendix A for their derivation).

$$u'(q_o) = 1 + i, \tag{2}$$
$$U'(x_o) = \mu. \tag{3}$$

Equation (2) is obtained from the envelope condition of an old agent in the goods market. It states that the opportunity cost of holding money, i, is equal to the benefit of spending the marginal unit for consumption in the goods market. Equation (3) represents the first-order condition with respect to x_o in the centralized market and states that the centralized market consumption of old agents is independent of monetary policy.

The equilibrium conditions of young agents depend on the type of equilibrium which prevails. In what follows, we assume that the critical labor effort, $\mathfrak{L}$, is sufficiently large, such that there is a meaningful difference between young and old agents. Concretely, we assume that $\mathfrak{L} > \mathfrak{L}_1(\mu - 1) := U'^{-1}(1) + u'^{-1}(1) - \phi T$ and find that the model features two types of equilibria, which we characterize hereafter.[8] All proofs are relegated to Appendix A.

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[8] In Appendix B, we show that when we remove this assumption, i.e., such that young and old agents do not differ much, the model features two additional equilibria, which we label type-III and type-IV equilibrium. In a type-III equilibrium, the cash constraint of a young buyer is binding in the goods market, and young agents work more than $\mathfrak{L} + \varepsilon$, i.e., $\ell_y > \mathfrak{L} + \varepsilon$. In a type-IV equilibrium, the cash constraint of a young buyer is binding in the goods market, and young agents work less than $\mathfrak{L} + \varepsilon$ and more than $\mathfrak{L} - \varepsilon$, i.e., $\mathfrak{L} - \varepsilon < \ell_y < \mathfrak{L} + \varepsilon$.

Type-I Equilibrium   In a type-I equilibrium, the cash constraint of a young buyer is non-binding in the goods market, and young agents work less than $\mathfrak{L} + \varepsilon$ and more than $\mathfrak{L} - \varepsilon$, i.e., $1 < f'(\ell_y) < \mu$. A type-I equilibrium is characterized by the following:

Proposition 1 *A type-I equilibrium is a list $\{q_o, q_y, x_o, x_y\}$ satisfying* (2), (3), *and*

$$
\begin{aligned}
u'(q_y) &= 1, & (4) \\
U'(x_y) &= \mu/(1+i). & (5)
\end{aligned}
$$

Equation (4) states that young buyers consume the efficient quantity q^* in the goods market, which solves $u'(q^*) = 1$. Equation (5) states that the consumption of young agents in the centralized market depends on monetary policy; i.e., the higher the opportunity cost of holding money becomes, the higher centralized market consumption will be. This is because the centralized market consumption of young agents is low in this equilibrium; i.e., we have $\mu/(1+i) = U'(x_y) = f'(\ell_y) \in (1, \mu)$ by the first-order condition. Hence, a type-I equilibrium does not exist if the opportunity cost of holding money is too large. This is so, because saving for the efficient goods market consumption is too costly when i is large. Under the Friedman rule, $i = 0$, the marginal disutility of labor of young agents is equal to that of old agents; i.e., $f'(\ell_y) = \mu$. However, if $i > 0$, then the marginal disutility of labor of young agents is strictly smaller than that of old agents; i.e., $f'(\ell_y) < \mu$. This means that if there is a positive opportunity cost of holding money, young agents optimally choose not to suffer the full marginal disutility of labor in contradiction to old agents. At the same time, they choose to consume more than old agents in the centralized market for any $i > 0$; i.e., we have $x_y > x_o$. There are three unique features of the type-I equilibrium:

- First, young agents save money so that they are prepared not only for their idiosyncratic preference shocks in the goods market, but also for their retirement.

- Second, young agents work more but consume fewer centralized market goods in response to low nominal interest rates, since $\mu/(1+i) = U'(x_y) = f'(\ell_y)$. This is because young agents prefer to benefit from their lower disutility of labor and the small opportunity cost of holding money. As a result, it could be possible that agents only work when they are young. This would happen, for instance, if $\mathfrak{L}$ is sufficiently large.

- Third, young agents consume the efficient quantity q^*, but old agents consume less than q^*; that is, old agents only consume q_o in the goods market, where $u'(q_o) = 1 + i$. In the type-I equilibrium, young agents may be able to save enough money to be able to consume q^* in the goods market when they are old (for instance, when $\mathfrak{L}$ is very large). However, old agents do not want to consume q^*, simply because it is not optimal for them. When old, time becomes a more important factor in their decisions. As a result, old agents would prefer to buy more leisure in the centralized market rather than to consume additional goods $q > q_o$ in the goods market. More precisely, suppose that an old agent has additional resources to use. The marginal utility of an additional unit of leisure is $\mu > 1$. However,

the marginal utility of an additional unit of consumption, $q > q_o$, in the goods market is $u'(q)/(\phi p) < u'(q_o)/(\phi p) = \mu$, where $\phi p = (1+i)/\mu$ is the relative price of the specialized good with respect to leisure. Therefore, in the type-I equilibrium, young agents consume the efficient quantity q^* in the goods market, but old agents optimally choose to consume less than q^*.

Type-II Equilibrium   In a type-II equilibrium, the cash constraint of a young buyer is binding in the goods market, and young agents work less than $\mathfrak{L}-\varepsilon$, i.e., $\ell_y < \mathfrak{L}-\varepsilon$. Hence, the marginal disutility of labor of young agents equals 1, $f'(\ell_y) = 1$. A type-II equilibrium is characterized by the following:

Proposition 2 *A type-II equilibrium is a list $\{q_o, q_y, x_o, x_y\}$ satisfying (2), (3), and*

$$u'(q_y) = (1+i)/(n\mu) - (1-n)/n, \qquad (6)$$
$$U'(x_y) = 1. \qquad (7)$$

Equation (6) is obtained by rearranging the envelope condition of a young agent entering the goods market. It states that the consumption of young agents is inefficiently low in the goods market. On the contrary, equation (7) shows that the centralized market consumption of young agents is efficient and independent of monetary policy, $U'(x_y) = f'(\ell_y) = 1$. Since $u(q_y) > 1$, equation (6) implies that $\mu/(1+i) < 1$. Hence, a type-II equilibrium exists only if i is large enough. In this case, young agents optimally choose to consume more goods in the centralized market by reducing their savings for the goods market. In addition, $f'(\ell_y) = 1 < \mu$ implies that young agents optimally choose to minimize the marginal disutility of labor.

4.1   Regions of Existence

We now describe the regions of existence of each type of equilibrium. The following proposition characterizes two non-overlapping regions in which these two types of equilibria exist.

Proposition 3 *If $i < \mu - 1$, equilibrium prices and quantities are characterized by Proposition 1; and if $i > \mu - 1$, they are characterized by Proposition 2.*

TABLE 1: CHARACTERISTICS OF EQUILIBRIA

Equilibrium	q_y	x_y	ℓ_y
Type-I	$q_y = q^*$	$U'(x_y) = \mu/(1+i)$	$\mathfrak{L}-\varepsilon < \ell_y < \mathfrak{L}+\varepsilon$
Type-II	$q_y < q^*$	$U'(x_y) = 1$	$\ell_y < \mathfrak{L} - \varepsilon$

Table 1 presents the key characteristics of each equilibrium. If the nominal interest rate is low enough, $i < \mu - 1$, then the type-I equilibrium exists. Since the opportunity cost of holding money is small in the type-I equilibrium, young agents save money by consuming less in the centralized market, $U'(x_y) \geq 1$. Because young agents save enough money, they can consume

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the optimal amount of goods $q_y = q^*$ if they become buyers. In this type of equilibrium, the opportunity cost of holding money is small, such that it is suboptimal to work less than $\mathfrak{L} - \varepsilon$. Since the marginal disutility of labor would be too high, $f'(\ell_y) = \mu$ for $\ell_y > \mathfrak{L} + \varepsilon$, young agents choose to work in the range $\mathfrak{L} - \varepsilon < \ell_y < \mathfrak{L} + \varepsilon$ in the type-I equilibrium. Equivalently, $1 < f'(\ell_y) < \mu$. As a result, young agents can still consume more centralized market goods than old agents, because the marginal disutility of labor of young agents is lower than that of old agents, μ. Therefore, we have $x_y \geq x_o$.

For higher nominal interest rates, $i > \mu - 1$, the type-II equilibrium exists. Since the opportunity cost of holding money is high, young agents prefer to consume more centralized market goods, $U'(x_y) = 1$, by saving less for the goods market. Since they save less, they consume less, $q_y < q^*$, in the goods market. A high opportunity cost of holding money also results in less work, $\ell_y < \mathfrak{L} - \varepsilon$ with $f'(\ell_y) = 1$, in this type of equilibrium.

5 Monetary Policy

We now derive the effect on consumption in each market, x and q, in response to a marginal increase in the exogenous policy parameter i in each equilibrium.

Proposition 4 *In any type of equilibrium,*

$$\frac{\partial x_o}{\partial i} = 0, \qquad \frac{\partial q_o}{\partial i} < 0.$$

The general goods consumption of old agents is independent of the nominal interest rate i, $U'(x_o) = \mu$. Consequently, we have $\partial x_o / \partial i = 0$. Since $u'(q_o) = 1 + i$, old agents consume less in the goods market ($\partial q_o / \partial i < 0$) when i increases.

Proposition 5 *In the type-I equilibrium,*

$$\frac{\partial x_y}{\partial i} > 0, \qquad \frac{\partial q_y}{\partial i} = 0.$$

In the type-I equilibrium, young agents consume the optimal quantity of specialized goods, $u'(q^*) = 1$, and thus $\partial q_y / \partial i = 0$. Since the opportunity cost of holding money is low in the type-I equilibrium, young agents consume a low amount of general goods in the centralized market and save money in order to prepare their retirement. As the opportunity cost of holding money increases, young agents have a lower incentive to save, so they increase their consumption of general goods in the centralized market ($\partial x_y / \partial i > 0$) when i increases.

The relationship between the consumption of general goods, labor, and savings is graphically depicted in Figure 3. The figure shows that young agents save the difference between ℓ_y and x_y for their retirement; i.e., given the budget constraint in the centralized market, we have $\ell_y - x_y = \phi m_y - \phi T$.
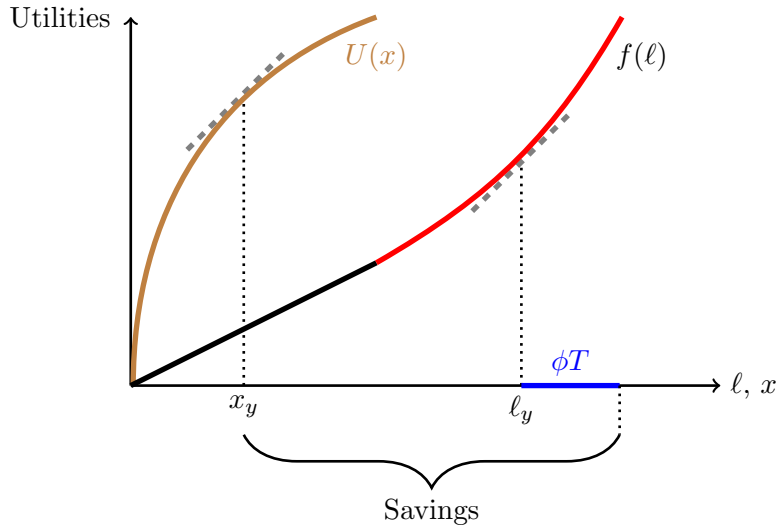
FIGURE 3: SAVINGS IN THE TYPE-I EQUILIBRIUM

Proposition 6 *In the type-II equilibrium,*

$$\frac{\partial x_y}{\partial i} = 0, \qquad \frac{\partial q_y}{\partial i} < 0.$$

In the type-II equilibrium, the opportunity cost of holding money is high. Hence, young agents consume a higher amount of general goods in the centralized market, $U'(x_y) = 1$, but they cannot consume the optimal amount of specialized goods in the goods market, $q_y < q^*$. As the opportunity cost of holding money increases, they consume fewer specialized goods in the goods market ($\partial q_y / \partial i < 0$).

It is worth noting that increasing inflation hurts old agents, but benefits young agents in standard overlapping generations models (e.g., Grandmont, 1985). However, we find that this result does not hold anymore in our model when the inflation rate exceeds a certain threshold. In the type-II equilibrium, increasing inflation lowers consumption for both young and old agents. That is, young and old agents do not change the level of their consumption of the general good, but instead reduce their consumption of the specialized good. As a result, increasing inflation hurts both young and old agents in the type-II equilibrium.

The relationship between the consumption of general goods, labor, and savings is graphically depicted in Figure 4. Since ℓ_y is not determined by $U'(x_y) = f'(\ell_y) = 1$ in the type-II equilibrium, savings cannot be found by means of the budget constraint in the centralized market. Therefore, savings are determined by
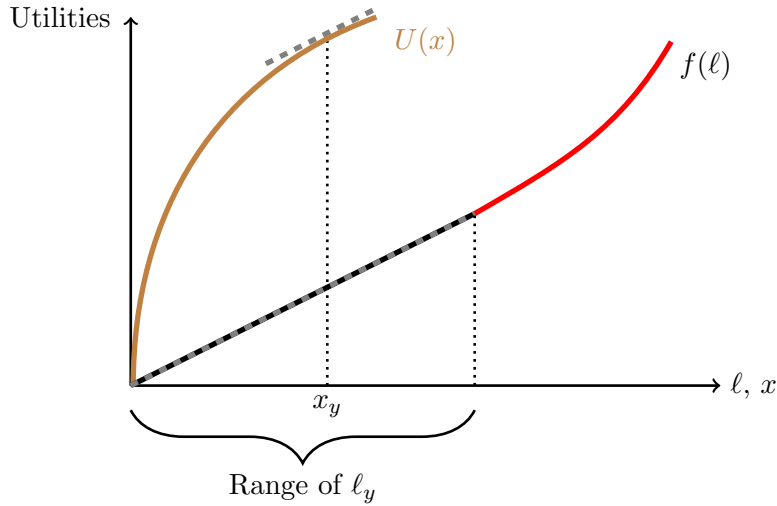
$$u'(m_y/p) = u'(q_y) = (1+i)/n\mu - (1-n)/n,$$

FIGURE 4: RANGE OF ℓ_y IN THE TYPE-II EQUILIBRIUM

and ℓ_y is determined by

$$x_y + \phi m_y = \ell_y + \phi T;$$

i.e., agents only work and save money for their consumption in the period, but not for their retirement.

5.1   Welfare

Here, we explain why increasing the exogenous policy parameter i can be welfare-improving. The reason is straightforward. In the type-I equilibrium, centralized market consumption is low, because young agents prefer to save money for their retirement. Increasing i does not affect the goods market consumption of young agents, since they can still consume the optimal amount of goods $q_y = q^*$. However, it helps to increase the centralized market consumption of young agents, because saving becomes less attractive. The drawback of increasing i is that old agents consume less in the goods market. We show that for low values of i, the former effect dominates the latter, such that it is welfare-improving to increase inflation above the Friedman rule.

We follow the literature that builds on Lagos and Wright (2005) and define welfare as the sum of all agents' utilities and dis-utilities from consumption and production of goods. Using the market clearing condition in the goods market, $nq_y + q_o = (1 - n)q^s$, the welfare function can be written as follows:

$$(1 - \beta)\mathcal{W} = [U(x_y) - f(x_y) + nu(q_y) - (1 - n)q^s] + [U(x_o) - \mu x_o + u(q_o)]$$
$$= [U(x_y) - f(x_y) + nu(q_y) - nq_y] + [U(x_o) - \mu x_o + u(q_o) - q_o].$$

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Hence, we can derive

$$(1 - \beta)\frac{\partial \mathcal{W}}{\partial i} = \left[U'(x_y) - f'(x_y) \right] \frac{\partial x_y}{\partial i} + n \left[u'(q_y) - 1 \right] \frac{\partial q_y}{\partial i} + \left[u'(q_o) - 1 \right] \frac{\partial q_o}{\partial i}. \qquad (8)$$

Proposition 7 *It holds that*

$$\frac{\partial \mathcal{W}(0)}{\partial i} > 0, \qquad and \qquad \frac{\partial \mathcal{W}(i)}{\partial i} < 0 \;\; for \;\; i > \mu - 1.$$

If i is sufficiently small, then the economy is in the type-I equilibrium where the level of general goods consumption is low. Especially, at $i = 0$, the consumption of general goods is so low that the marginal utility gain from increasing i is significant. If i is larger than $\mu - 1$, then the economy is in the type-II equilibrium, where young agents save too little for the goods market. As a result, increasing i is welfare-decreasing, since it decreases the goods market consumption of young and old agents, $\partial q_y/\partial i < 0$ and $\partial q_o/\partial i < 0$, while keeping centralized market consumption unchanged, $\partial x_y/\partial i = 0$. This finding implies that the welfare-maximizing nominal interest rate cannot exceed a certain level. Note that the lump-sum transfer ϕT is not included in equation (8); i.e., the welfare-improving role of inflation has nothing to do with the distributional effect of inflation, which is found in related studies which use overlapping generations models.

6   Discussion

Grandmont's (1985) seminal contribution on overlapping generations models is also related to our study. In Grandmont's model, there is a centralized market in each period. Further, he assumes that the marginal utility of young agents is smaller than that of old agents, which guarantees positive savings when agents are young. Grandmont (1985) also finds that output may increase with the inflation rate. However, this result requires a few additional assumptions. First, goods and leisure are assumed to be non-inferior commodities. Second, the marginal propensity to consume leisure is assumed to be higher for old agents than for young agents under any allocation of the real wealth distribution between young and old agents. However, the effects on welfare still remain puzzling under these assumptions.

Compared with Grandmont (1985), our condition concerning the disutility of labor implies his assumption that the marginal utility of young agents is smaller than that of old agents, which is why young agents save for their retirement. In contrast to his study, we do not require any further assumptions in order to obtain an association between increasing centralized market consumption and inflation. This is an important difference, since his additional assumptions are quite strong. That is, his latter assumption is equivalent to assuming that the marginal consumption with respect to real wealth is smaller for old agents than for young agents, independent of their real wealth.[9] It should therefore be clear that our mechanism is very different

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[9]In Grandmont (1985), the budget constraint is given by $c_\tau(a_\tau) + l_\tau^* - l_\tau(a_\tau) = a_\tau$, where consumption $c_\tau(a_\tau)$ and labor $l_\tau(a_\tau)$ are functions of the real wealth a_τ in each period τ, and l_τ^* denotes the endowment of labor.

from Grandmont's approach, where his assumptions drive the results to a large extent. In Table 2, we summarize the main differences between our model and Grandmont's study.

TABLE 2: COMPARISON WITH GRANDMONT AND LW

| Description | Our model | | Grandmont | LW |
	Low inflation	High inflation		
Saving for retirement	yes	no	yes	no
CM consumption in i	increase	constant	increase[†]	constant
GM consumption in i	decrease	decrease	—	decrease
Welfare in i	increase	decrease	—	decrease

† Grandmont (1985) imposes a few additional assumptions on consumption behaviors.

Additionally, in Table 2 we compare our model with Lagos and Wright (2005). Although we only add one additional assumption to their basic framework, our results are completely different from theirs for low inflation rates. Namely, in our model agents save for their retirement, and centralized market consumption is affected by monetary policy. Some other relationships are the same in both models; for example, agents reduce their goods market consumption as inflation increases. For high inflation rates, our model produces similar results as Lagos and Wright (2005), and hence, also welfare is decreasing in the inflation rate.

7   Conclusion

We integrate an overlapping generations model into a new monetarist framework and show that it is welfare-improving to increase inflation above the Friedman rule. This is because inflation makes saving for retirement less attractive, with the result that young agents optimally choose to increase their consumption, while old agents consume less due to the inflation tax. We show that for low inflation rates the former effect dominates the latter, such that it is optimal to increase inflation above the Friedman rule. However, in contrast to standard overlapping generations models, we find that higher inflation rates induce both young and old agents to reduce their consumption. Consequently, the optimal inflation rate must remain beneath a certain threshold level.

By differentiating the budget constraint with respect to real wealth a_τ, we can derive $c'_\tau(a_\tau) - l'_\tau(a_\tau) = 1$; i.e., $|l'_\tau(a_\tau)| = -l'_\tau(a_\tau) = 1 - c'_\tau(a_\tau)$. Therefore, Grandmont's assumption on $|l'_\tau(a_\tau)|$ is equivalent to an assumption on $c'_\tau(a_\tau)$; i.e., $|l'_2(a_2)| > |l'_1(a_1)|$ for any $a_1 + a_2 = l_1^* + l_2^*$ if, and only if, $c'_2(a_2) < c'_1(a_1)$ for any $a_1 + a_2 = l_1^* + l_2^*$. Especially, $c'_2(\varepsilon) < c'_1(l_1^* + l_2^* - \varepsilon)$ for small ε might be a strong assumption by considering that $c_2(0) = 0$, and utility is strictly concave in consumption.

A Appendix: Proofs

We first derive the solutions to the decision problems of an agent of generation t.

In period $t+1$: In period $t+1$, an agent of generation t is old. The envelope condition of an old agent in the goods market equals

$$\partial W_o(\phi_{+1} m_{o,+1})/\partial m_{o,+1} = u'(q_{o,+1})/p_{+1}. \tag{9}$$

The above equation simply states that the marginal utility of money for an old agent equals the utility gain from spending the marginal unit on consumption.

The first-order conditions of the centralized market of an old agent with respect to $m_{o,+1}$ and $x_{o,+1}$ are

$$\partial W_o(\phi_{+1} m_{o,+1})/\partial m_{o,+1} = \mu\phi_{+1}, \qquad \text{and} \qquad U'(x_{o,+1}) = \mu. \tag{10}$$

The first-order condition with respect to $x_{o,+1}$ shows that old agents consume the efficient quantity $x_{o,+1}^*$ in the centralized market, which solves $U'(x_{o,+1}^*) = \mu$. The term $\partial W_o(\phi_{+1} m_{o,+1})$ $/\partial m_{o,+1}$ denotes the marginal benefit from bringing one additional unit of money into the goods market and $\mu\phi_{+1}$ is the marginal cost of doing so. The envelope condition equals

$$\partial V_o(\phi_{+1}\hat{m}_{o,+1})/\partial\hat{m}_{o,+1} = \mu\phi_{+1}. \tag{11}$$

Equation (11) states that the marginal value of money at the beginning of the centralized market equals the price of money in terms of general goods times the marginal disutility of labor of an old agent.

In period t: In period t, an agent of generation t is young. The first-order condition of a young seller in the goods market equals

$$1 = \beta p \partial V_o(\phi_{+1}\hat{m}_{o,+1})/\partial\hat{m}_{o,+1} = \beta p \mu\phi_{+1}; \tag{12}$$

i.e., the marginal cost of producing q^s units of specialized goods ($c'(q^s) = 1$) equals the price p times the discounted marginal cost of acquiring an additional unit of money in the next period's centralized market. Hence, a young seller is willing to produce the quantity q^s that clears the market. A young seller's envelope condition equals

$$\partial W_y^s(\phi m_y)/\partial m_y = \beta\partial V_o(\phi_{+1}\hat{m}_{o,+1})/\partial\hat{m}_{o,+1} = \beta\mu\phi_{+1}. \tag{13}$$

The envelope condition states that a young seller's marginal utility of money equals the marginal value of money at the beginning of the centralized market. This is simply because money has no value to him in the goods market.

The first-order condition and the envelope condition of a young buyer depend on whether his cash constraint is binding or not. If the cash constraint is non-binding, a young buyer is

able to consume the efficient quantity, such that

$$u'(q_y) = \beta p \partial V_o(\phi_{+1}\hat{m}_{o,+1})/\partial \hat{m}_{o,+1} = \beta p \mu \phi_{+1} = 1. \tag{14}$$

In this case, the envelope condition equals

$$\partial W_y^b(\phi m_y)/\partial m_y = \beta \partial V_o(\phi_{+1}\hat{m}_{o,+1})/\partial \hat{m}_{o,+1} = \beta \mu \phi_{+1}. \tag{15}$$

The above equation simply states that the marginal benefit from bringing an additional unit of money into the next period, $\partial W_y^b(\phi_t m_{y,t})/\partial m_{y,t}$, equals the discounted marginal cost of acquiring it in the next period's centralized market.

If the cash constraint is binding, we have

$$pq_y = m_y, \qquad \text{and hence} \qquad W_y^b(\phi m_y) = u(q_y) + \beta V_o(0). \tag{16}$$

Consequently, the envelope condition equals

$$\partial W_y^b(\phi m_y)/\partial m_y = u'(q_y)/p. \tag{17}$$

That is, the marginal benefit of money equals the utility of spending the marginal unit on consumption in the goods market.

Finally, the envelope condition of a young agent entering the goods market equals

$$\partial W_y(\phi m_y)/\partial m_y = \beta \mu \phi_{+1} = 1/p, \tag{18}$$

if a young buyer's cash constraint is non-binding, and

$$\partial W_y(\phi m_y)/\partial m_y = nu'(q_y)/p + (1-n)/p, \tag{19}$$

if the cash constraint is binding.

The first-order conditions with respect to m_y and x_y of a young agent in the centralized market are

$$\partial W_y(\phi m_y)/\partial m_y = \phi f'(\ell_y), \qquad \text{and} \qquad U'(x_y) = f'(\ell_y). \tag{20}$$

The above equations have a similar interpretation as (10).

Derivation of (2). Using the equations (1), (9), (10), (12), we have

$$u'(q_{o,+1}) = 1 + i_{+1}.$$

By simply updating the period backwards, we obtain (2).

Derivation of (3). Equation (3) can be obtained from the first-order condition (10) by updating period backward.

Proof of Proposition 1. *Derivation of (4).* Equation (4) equals the first-order conditions of a young buyer in the goods market when the cash constraint is non-binding, (14).

Derivation of (5). From equation (20), we obtain $\partial W_y(\phi m_y)/\partial m_y = \phi U'(x_y)$. Since a young

buyer's cash constraint is non-binding in the goods market, the envelope condition of a young agent entering the goods market is given by equation (18). Therefore, we have $U'(x_y) = 1/(p\phi)$. Using the equations (1) and (12), we obtain equation (5). ∎

Proof of Proposition 2. *Derivation of (6).* For a binding cash constraint, the envelope condition of a young agent entering the goods market equals (19). Using this together with equation (20) and the equilibrium condition $f'(\ell_y) = 1$ to replace $\partial W_y(\phi m_y)/\partial m_y$, we obtain

$$-\phi + \left[n u'(q_y) + 1 - n \right]/p = 0.$$

Using (1) and (12) to replace p, and simplifying terms, yields (*6*).

Derivation of (7). Equation (7) represents the first-order condition (20) in the centralized market of a young agent under the equilibrium condition $f'(\ell_y) = 1$. ∎

Proof of Proposition 3. The critical value that separates the type-I from the type-II equilibrium requires that (5) and (7) hold simultaneously. Simplifying terms yields $i = \mu - 1$. ∎

Proof of Proposition 4. $\partial x_o/\partial i$ and $\partial q_o/\partial i$ are directly obtained from equations (2) and (3). That is, we obtain

$$\frac{\partial x_o}{\partial i} = 0, \qquad \frac{\partial q_o}{\partial i} = \frac{1}{u''[u'^{-1}(1+i)]} < 0.$$

∎

Proof of Proposition 5. $\partial x_y/\partial i$ and $\partial q_y/\partial i$ are directly obtained from equations (4) and (5). That is, we obtain

$$\frac{\partial x_y}{\partial i} = -\frac{\mu}{(1+i)^2 U''(U'^{-1}[\mu/(1+i)])} > 0, \qquad \frac{\partial q_y}{\partial i} = 0.$$

∎

Proof of Proposition 6. $\partial x_y/\partial i$ and $\partial q_y/\partial i$ are directly obtained from equations (6) and (7). That is, we obtain

$$\frac{\partial x_y}{\partial i} = 0, \qquad \frac{\partial q_y}{\partial i} = \frac{1}{n\mu u''(u'^{-1}[(1+i)/(n\mu) - (1-n)/n])} < 0.$$

∎

Proof of Proposition 7. *Derivation of statement 1.* From Proposition 5 and equation (8), we can derive

$$(1-\beta)\frac{\partial \mathcal{W}}{\partial i} = -\frac{\mu \left[U'(x_y) - f'(x_y) \right]}{(1+i)^2 U''(x_y)} + \frac{i}{u''(q_o)}.$$

At $i = 0$, there is no lump-sum transfer $\phi T = 0$, which implies $x_y < \ell_y$ by the budget constraint,

$x_y + \phi m_y = \ell_y$ with $\phi m_y > 0$. Hence, we can derive

$$U'(x_y) - f'(\ell_y) = 0 < U'(x_y) - f'(x_y),$$

because f is convex and $f''(\ell_y) > 0$. Therefore, the first term of $(1-\beta)\partial\mathcal{W}/\partial i$ is positive at $i = 0$ and the second term vanishes at $i = 0$. As a result, we have

$$\frac{\partial\mathcal{W}(0)}{\partial i} > 0.$$

Derivation of statement 2. From Proposition 6 and equation (8), we can derive

$$(1-\beta)\frac{\partial\mathcal{W}}{\partial i} = \frac{1+i-\mu}{n\mu^2 u''(u'^{-1}[(1+i)/(n\mu) - (1-n)/n])} + \frac{i}{u''[u'^{-1}(1+i)]} < 0.$$

where both terms are non-positive and cannot be zero at the same time. Therefore, welfare is strictly decreasing in i. ∎

B   Appendix: Removing the Assumption $\mathfrak{L} > \mathfrak{L}_1(\mu - 1)$

B.1   Two Additional Equilibria

When we eliminate the assumption of $\mathfrak{L} > \mathfrak{L}_1(\mu - 1)$, we find that the model features two additional equilibria, which we label type-III and type-IV.

Type-III Equilibrium   In a type-III equilibrium, the cash constraint of a young buyer is binding in the goods market, and young agents work more than $\mathfrak{L} + \varepsilon$, i.e., $\ell_y > \mathfrak{L} + \varepsilon$. Hence, the marginal disutility of labor of young agents equals μ, $f'(\ell_y) = \mu$. A type-III equilibrium is characterized by the following:

Proposition 8 *A type-III equilibrium is a list $\{q_o, q_y, x_o, x_y\}$ satisfying (2), (3), and*

$$\begin{align}
u'(q_y) &= (1+i)/n - (1-n)/n, \tag{21}\\
U'(x_y) &= \mu. \tag{22}
\end{align}$$

Proof of Proposition 8.   *Derivation of (21).* For a binding cash constraint, the envelope condition of a young agent entering the goods market equals (19). Together with equation (20) and the equilibrium condition $f'(\ell_y) = \mu$ to replace $\partial W_y(\phi m_y)/\partial m_y$, we obtain

$$-\mu\phi + \left[nu'(q_y) + 1 - n\right]/p = 0.$$

Using (1) and (12) to replace p, and simplifying terms, yields (21).

Derivation of (22). Equation (22) represents the first-order condition (20) in the centralized market of a young agent under the equilibrium condition $f'(\ell_y) = \mu$. ∎

Equation (21) and (22) have a similar interpretation to their counterparts in Proposition 2. However, contrary to the type-I and the type-II equilibrium, a type-III equilibrium can exist for any rate of $i \geq 0$, because $u'(q_y) = (1 + i)/n - (1 - n)/n \geq 1$ for any $i \geq 0$, and the centralized market consumption of young agents is independent of i from equation (22). A type-III equilibrium is characterized by a low level of consumption in both markets in comparison with the type-I and the type-II equilibrium, because $\mu > \max\{\mu/(1 + i), 1\}$ and $(1 + i)/n - (1 - n)/n > \max\{(1 + i)/(n\mu) - (1 - n)/n, 1\}$ for any positive $i > 0$. Note that, in a type-III equilibrium, the marginal disutility of labor of young agents is equal to the marginal disutility of labor of old agents, $f'(\ell_y) = \mu$. This is because consumption in both markets is low, such that young agents endure the full marginal disutility of labor for any $i \geq 0$ in a type-III equilibrium. It is worth mentioning that the type-III equilibrium is an equilibrium that is analyzed in many other studies that build on Lagos and Wright (2005), where the marginal disutility of labor of all agents is identical and constant over time.

Type-IV Equilibrium In a type-IV equilibrium, the cash constraint of a young buyer is binding in the goods market, and young agents work less than $\mathfrak{L} + \varepsilon$ and more than $\mathfrak{L} - \varepsilon$, i.e., $\mathfrak{L} - \varepsilon < \ell_y < \mathfrak{L} + \varepsilon$. A type-IV equilibrium is characterized by the following

Proposition 9 *A type-IV equilibrium is a list* $\{q_o, q_y, x_o, x_y\}$ *satisfying* (2), (3), *and*

$$
x_y = f'^{-1}\left[U'(x_y)\right] - q_y(1 + i)/\mu + \phi T, \tag{23}
$$
$$
U'(x_y) = \left\{n\left[u'(q_y) - 1\right] + 1\right\}\mu/(1 + i). \tag{24}
$$

Proof of Proposition 9. *Derivation of (23).* From equation (20), we obtain $\ell_y = f'^{-1}(U'(x_y))$. Since the cash constraint is binding, we have (16) and consequently $pq_y = m_y$. Using equations (1) and (12), we obtain $\phi m_y = \phi p q_y = q_y(1 + i)/\mu$. Together with the budget constraint of a young agent entering the goods market,

$$
x_y + \phi m_y = \ell_y + \phi T,
$$

we obtain (23).

Derivation of (24). For a binding cash constraint, the envelope condition of a young agent entering the goods market equals (19). By replacing $\partial W_y(\phi m_y)/\partial m_y$ in equation (20), we obtain $\phi U'(x_y) = nu'(q_y)/p + (1 - n)/p$. Using (1) and (12) to replace p, and simplifying terms, yields (24). ∎

Equation (23) is obtained by rearranging the budget constraint of a young agent in the centralized market. As in the type-I equilibrium, the consumption of young agents in the centralized market increases with the nominal interest rate. However, (24) clearly shows that x_y is lower in the type-IV equilibrium than in the type-I equilibrium, since $u'(q_y) > 1$.

B.2 Regions of Existence

We now describe the regions of existence of each type of equilibrium. To do this, we consider a limiting case of f when ε approaches zero for simplicity of analysis. Note that the regions of existence of each type of equilibrium depend on the nominal interest rate i and the disutility function f. However, the function f cannot be described as a one dimensional parameter in general. It is worthwhile noting that the equilibrium conditions in the type-I, II, and III equilibria do not depend on the shape of the function f. Thus, this additional simplification will not change our results qualitatively.

Hence, we derive the regions of existence of each type of equilibrium with respect to the nominal interest rate i and the critical labor effort $\mathfrak{L}$. The following proposition guarantees the existence of each type of equilibrium.

Proposition 10 *There exist the functions $\mathfrak{L}_1(i)$, $\mathfrak{L}_2(i)$, $\mathfrak{L}_3(i)$, such that*

$$
\begin{array}{lll}
\mathfrak{L}_1 : [0, \mu - 1] \to \mathbb{R}, & \partial \mathfrak{L}_1(i)/\partial i > 0, & \\
\mathfrak{L}_2 : [\mu - 1, \infty) \to \mathbb{R}, & \partial \mathfrak{L}_2(i)/\partial i < 0, & \mathfrak{L}_2(\mu - 1) = \mathfrak{L}_1(\mu - 1), \\
\mathfrak{L}_3 : [0, \infty) \to \mathbb{R}, & \partial \mathfrak{L}_3(i)/\partial i < 0, & \mathfrak{L}_3(0) = \mathfrak{L}_1(0).
\end{array}
$$

Furthermore,

- *the type-I equilibrium is supported if and only if $i < \mu - 1$ and $\mathfrak{L} > \mathfrak{L}_1(i)$,*

- *the type-II equilibrium is supported if and only if $i > \mu - 1$ and $\mathfrak{L} > \mathfrak{L}_2(i)$,*

- *the type-III equilibrium is supported if and only if $\mathfrak{L} < \mathfrak{L}_3(i)$,*

- *the type-IV equilibrium is supported if and only if (i) $i < \mu - 1$ and $\mathfrak{L}_3(i) < \mathfrak{L} < \mathfrak{L}_1(i)$; or (ii) $i > \mu - 1$ and $\mathfrak{L}_3(i) < \mathfrak{L} < \mathfrak{L}_2(i)$.*

Proof of Proposition 10. *Derivation of $i = \mu - 1$.* The critical value that separates the type-I from the type-II equilibrium requires that (5) and (7) hold simultaneously. Simplifying terms yields $i = \mu - 1$.

Derivation of $\mathfrak{L}_1(i)$. The function $\mathfrak{L}_1(i)$ separates the type-I from the type-IV equilibrium. It requires that (4), (5), (23), and $f'^{-1}[U'(x_y)] = \ell_y = \mathfrak{L}_1(i)$ hold simultaneously. Simplifying terms yields

$$
\mathfrak{L}_1(i) = U'^{-1}[\mu/(1 + i)] + u'^{-1}(1)(1 + i)/\mu - \phi T.
$$

Furthermore,

$$
\frac{\partial \mathfrak{L}_1(i)}{\partial i} = -\frac{\mu}{(1 + i)^2} \frac{1}{U''(U'^{-1}[\mu/(1 + i)])} + \frac{u'^{-1}(1)}{\mu} > 0.
$$

Derivation of $\mathfrak{L}_2(i)$. The function $\mathfrak{L}_2(i)$ separates the type-II from the type-IV equilibrium. It requires that (6), (7), (23), and $f'^{-1}[U'(x_y)] = \ell_y = \mathfrak{L}_2(i)$ hold simultaneously. Simplifying

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terms yields

$$\mathfrak{L}_2(i) = U'^{-1}(1) + u'^{-1}\left[(1+i)/(n\mu) - (1-n)/n\right](1+i)/\mu - \phi T.$$

Furthermore,

$$\frac{\partial \mathfrak{L}_2(i)}{\partial i} = \frac{1+i}{n\mu^2}\frac{1}{u''(q_y)} + \frac{q_y}{\mu} < \frac{1}{\mu}\left[\frac{u'(q_y)}{u''(q_y)} + q_y\right] = \frac{1}{\mu}\left[-\frac{q_y}{\alpha(q_y)} + q_y\right] \leq 0,$$

and

$$\mathfrak{L}_1(\mu - 1) = \mathfrak{L}_2(\mu - 1).$$

Derivation of $\mathfrak{L}_3(i)$. The function $\mathfrak{L}_3(i)$ separates the type-III from the type-IV equilibrium. It requires that (21), (22), (23), and $f'^{-1}\left[U'(x_y)\right] = \ell_y = \mathfrak{L}_3(i)$ hold simultaneously. Simplifying terms yields

$$\mathfrak{L}_3(i) = U'^{-1}(\mu) + u'^{-1}\left[(1+i)/n - (1-n)/n\right](1+i)/\mu - \phi T.$$

Furthermore,

$$\frac{\partial \mathfrak{L}_3(i)}{\partial i} = \frac{1+i}{n\mu}\frac{1}{u''(q_y)} + \frac{q_y}{\mu} < \frac{1}{\mu}\left[\frac{u'(q_y)}{u''(q_y)} + q_y\right] \leq 0,$$

and

$$\mathfrak{L}_1(0) = \mathfrak{L}_3(0), \qquad \text{and} \qquad \mathfrak{L}_2(i) > \mathfrak{L}_3(i) \text{ for any } i \geq \mu - 1.$$

■

Figure B.1 shows stylistically the regions of existence as a function of i and $\mathfrak{L}$, while Table B.1 presents the key characteristics of each equilibrium.

TABLE B.1: CHARACTERISTICS OF EQUILIBRIA WITH $\varepsilon = 0$

Equilibrium	q_y	x_y	ℓ_y
Type-I	$q_y = q^*$	$U'(x_y) = \mu/(1+i)$	$\ell_y = \mathfrak{L}$
Type-II	$q_y < q^*$	$U'(x_y) = 1$	$\ell_y < \mathfrak{L}$
Type-III	$q_y < q^*$	$U'(x_y) = \mu$	$\ell_y > \mathfrak{L}$
Type-IV	$q_y < q^*$	$U'(x_y) > 1$	$\ell_y = \mathfrak{L}$

Figure B.1 shows that for high values of the critical labor effort, $\mathfrak{L} > \mathfrak{L}_1(\mu - 1)$, the type-I equilibrium or the type-II equilibrium exist depending on the nominal interest rate. The key features of these two equilibria are described in the main text of this paper. For low values of the critical labor effort, $\mathfrak{L} < \mathfrak{L}_3(\infty)$, only the type-III equilibrium exists, independent of the nominal interest rate. In this equilibrium, young agents suffer the same disutility of labor as old agents and thus (i) consume the same amount $x_y = x_o$ of centralized market goods as old agents do, and (ii) are not able to consume the efficient quantity in the goods market, $q_y < q^*$. Because the consumption level of young agents is low in both markets, they prefer to work more than the given low value of the critical labor effort. Therefore, we have $\ell_y > \mathfrak{L}$ with $f'(\ell_y) = \mu$
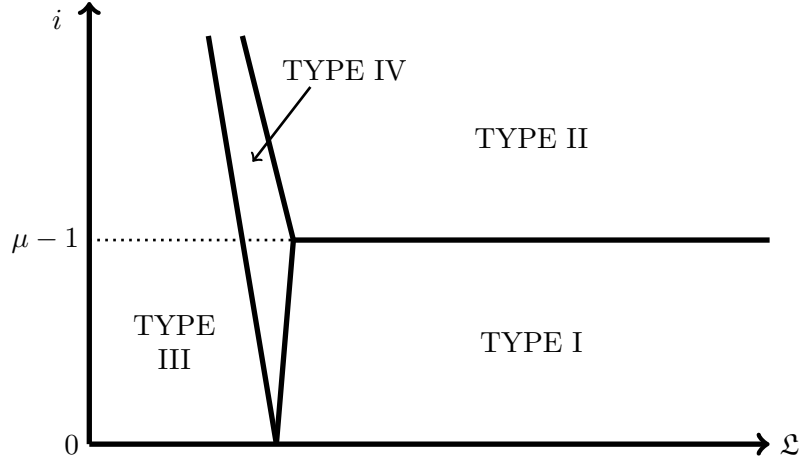
FIGURE B.1: REGIONS OF EXISTENCE WITH $\varepsilon = 0$

in the type-III equilibrium.

The type-IV equilibrium either exists between the type-III and the type-I equilibrium or between the type-III and the type-II equilibrium depending on i. The type-IV equilibrium shows mixed characteristics of the other types of equilibria. Concretely, we find that $U'(x_y) > 1$ in the centralized market similar to the type-I equilibrium, while we obtain $q_y < q^*$ similar to the type-II and the type-III equilibrium. Figure B.1 shows that for intermediate values of $\mathfrak{L}$, $\mathfrak{L}_3(\infty) < \mathfrak{L} < \mathfrak{L}_1(\mu-1)$, the sequence of equilibria is either (i) type-III and type-IV, (ii) type-III, type-IV, and type-II, or (iii) type-I, type-IV, and type-II as i increases from 0 to infinity.

B.3 Monetary Policy

We now derive the effect on the endogenous variables, x and q, in response to a marginal increase in the exogenous policy parameter i in the type-III equilibrium.

Proposition 11 *In the type-III equilibrium,*

$$\frac{\partial x_y}{\partial i} = 0, \qquad \frac{\partial q_y}{\partial i} < 0.$$

Proof of Proposition 11. $\partial x_y/\partial i$ and $\partial q_y/\partial i$ are directly obtained from equations (21) and (22). That is, we obtain

$$\frac{\partial x_y}{\partial i} = 0, \qquad \frac{\partial q_y}{\partial i} = \frac{1}{nu''(u'^{-1}[(1+i)/n - (1-n)/n])} < 0.$$

∎

In the type-III equilibrium, the disutility of labor of young agents is high compared to the other types of equilibria. Hence, young agents only consume the same amount of general goods

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as old agents do. As in the type-II equilibrium, young agents consume less specialized goods in the goods market ($\partial q_y / \partial i < 0$), while keeping the lowest level of general goods consumption ($\partial x_y / \partial i = 0$) when i increases.

B.4   Welfare

The implications concerning the optimal monetary policy are as follows in the type-III and the type-IV equilibrium.

Proposition 12 *In the type-III equilibrium, $\partial \mathcal{W} / \partial i < 0$.*

Proof of Proposition 12.   From Proposition 11 and equation (8), we can derive

$$(1 - \beta)\frac{\partial \mathcal{W}}{\partial i} = \frac{i}{nu''(u'^{-1}[(1+i)/n - (1-n)/n])} + \frac{i}{u''[u'^{-1}(1+i)]} \leq 0,$$

where both terms are strictly negative if $i > 0$. Therefore, welfare is strictly decreasing in $i > 0$. ∎

In the type-III equilibrium, $\mathfrak{L}$ is small such that young agents face the same marginal disutility of labor as old agents do. Proposition 12 confirms that the type-III equilibrium shows the standard characteristics of many other models that build on Lagos and Wright (2005).

The type-IV equilibrium shows mixed characteristics of the other types of equilibria.

References

Bhattacharya, J., Haslag, J., and Martin, A., 2005, "Heterogeneity, Redistributions, and the Friedman Rule", *International Economic Review*, 46, 437–454.

Berentsen, A., Camera, G., and Waller, C., 2005, "The Distribution of Money Balances and the Nonneutrality of Money", *International Economic Review*, 46, 465–487.

Berentsen, A., Rocheteau, G., and Shi, S., 2007, "Friedman Meets Hosios: Efficiency in Search Models of Money", *The Economic Journal*, 117, 174–195.

Bewley, T., 1980, "The Optimum Quantity of Money", *in Models of Monetary Economics*, ed. by John Kareken and Neil Wallace, Minnesota: Federal Reserve Bank.

Bewley, T., 1983, "A Difficulty with the Optimum Quantity of Money", *Econometrica*, 51, 1485–1504.

Chiu, J., and Molico, M., 2010, "Liquidity, Redistribution, and the Welfare Cost of Inflation", *Journal of Monetary Economics*, 57, 428–438.

Chiu, J., and Molico, M., 2011, "Uncertainty, Inflation, and Welfare", *Journal of Money, Credit and Banking*, 43, 487–512.

Chiu, J., and Molico, M., 2014, "Short-Run Dynamics in a Search-Theoretic Model of Monetary Exchange", *Working Paper.*

Cooley, T. F., and Quadrini, V., 2004, "Optimal Monetary Policy in a Phillips-Curve World", *Journal of Economic Theory*, 118, 174–208.

Deviatov, A., and Wallace, N., 2001, "Another Example in which Lump-sum Money Creation is Beneficial", *Advances in Macroeconomics 1*, Article 1.

Friedman, M.,1969, "The Optimum Quantity of Money", in M. Friedman *The Optimum Quantity of Money and Other Essays*, Aldine, Chicago, 1–50.

Gomis-Porqueras, P., and Sanches, D., 2013, "Optimal Monetary Policy in a Model of Money and Credit", *Journal of Money, Credit and Banking*, 45, 701–730.

Grandmont, J. M., 1985, "On Endogenous Competitive Business Cycles", *Econometrica*, 53, 995–1045.

Kehoe, T. J., Levine, D. K., and Woodford, M., 1992, "The Optimum Quantity of Money Revisited", in P. Dasgupta et al., eds., *Economic Analysis of Market Games: Essays in Honor of Frank Hahn* (Cambridge, MA, and London: MIT Press), 501–526.

Lagos, R., and Wright, R., 2005, "A Unified Framework for Monetary Theory and Policy Evaluation", *Journal of Political Economy,* 113, 463–484.

Levine, D., 1991, "Asset Trading Mechanisms and Expansionary Policy", *Journal of Economic Theory*, 54, 148–164.

Menzio, G., Shi, S., and Sun, H., 2013, "A Monetary Theory with Non-Degenerate Distribution", *Journal of Economic Theory*, 148, 2266–2312.

Molico, M., 2006, "The Distribution of Money and Prices in Search Equilibrium", *International Economic Review,* 47, 701–722.

Rocheteau, G., and Rodriguez-Lopez, A., 2014, "Liquidity Provision, Interest Rates, and Unemployment", *Journal of Monetary Economics*, 65, 80–101.

Rocheteau, G., and Wright, R., 2005, "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium", *Econometrica*, 73, 175–202.

Rocheteau, G., and Wright, R., 2009, "Inflation and Welfare in Models with Trading Frictions", *in Monetary Policy in Low Inflation Economies*, ed. by Ed Nosal and Dave Altig, Cambridge University Press.

Rocheteau, G., Weill, P. O., and Wong, T. N., 2015a, "A Tractable Model of Monetary Exchange with Ex-Post Heterogeneity", *Working Paper.*

Rocheteau, G., Weill, P. O., and Wong, T. N., 2015b, "Working Through the Distribution: Money in the Short and Long Run", *Working Paper*.

Schreft, S., and Smith, B., 1997, "Money, Banking, and Capital Formation", *Journal of Economic Theory*, 73, 157–182.

Schreft, S., and Smith, B., 2002, "The Conduct of Monetary Policy with a Shrinking Stock of Government Debt", *Journal of Money, Credit and Banking*, 34, 848–882.

Shi, S., 1997, "A Divisible Search Model of Fiat Money", *Econometrica*, 65, 75–102.

Smith, B., 2002, "Monetary Policy, Banking Crises, and the Friedman Rule", *American Economic Review*, 92, 128–134.

Williamson, S., 2006, "Search, Limited Participation, and Monetary Policy", *International Economic Review,* 47, 107–128.

Zhu, T., 2008, "An Overlapping-Generations Model with Search", *Journal of Economic Theory*, 142, 318–331.