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Ordinal Potentials in Smooth Games

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Abstract. A non-cooperative n-player game is called *smooth* if strategy spaces are non-degenerate compact intervals and payoff functions are twice continuously differentiable. In the class of smooth games, exact potential games are known to admit a convenient characterization in terms of cross-derivatives (Monderer and Shapley, 1996a). This characterization extends easily to weighted potential games. However, no analogous characterization is known for ordinal potential games. The present paper derives simple necessary conditions for a smooth game to admit an ordinal potential. First, any ordinal potential game must exhibit pairwise strategic complements or substitutes at any interior equilibrium. Second, in games with more than two players, a condition is obtained on the (modified) Jacobian at any interior equilibrium. Taken together, these conditions are shown to correspond to a local analogue of the Monderer-Shapley condition for weighted potential games. We identify two classes of economic games for which our necessary conditions are also sufficient.

Keywords. Ordinal potentials · smooth games · strategic complements and substitutes · semipositive matrices

JEL-Codes. C6 Mathematical Methods · Programming Models · Mathematical and Simulation Modeling; C72 – Noncooperative Games; D43 Oligopoly and Other Forms of Market Imperfection; D72 Political Processes: Rent-Seeking, Lobbying, Elections, Legislatures, and Voting Behavior

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1. Introduction

In a potential game (Rosenthal, 1973; Monderer and Shapley, 1996a), players' preferences may be summarized in a single common objective function.[1] Knowing if a specific game admits a potential can be quite valuable. For example, the existence of a potential reduces the problem of finding a Nash equilibrium to a straightforward optimization problem. Therefore, it is of some interest to know the conditions under which a potential exists. Clearly, sufficient conditions are most desirable. However, necessary conditions are also important. After all, such conditions may help avoiding a futile search for a potential. Moreover, as will be shown, necessary conditions may be indicative of sufficient conditions as well.[2]

This paper considers *smooth* games, i.e., non-cooperative n-player games with the property that strategy spaces are non-degenerate compact intervals and payoff functions are twice continuously differentiable (Vives, 1999). In the class of smooth games, we derive simple necessary conditions for the existence of a generalized ordinal potential.[3] Certainly, a generalized ordinal potential cannot exist if it is possible to construct a *strict improvement cycle* (Voorneveld, 1997), i.e., a finite circular sequence of strategy profiles with the property that moving to the next profile in the sequence amounts to one player strictly raising her payoff by a unilateral change in strategy. However, in general, identifying strict improvement cycles in smooth games is not straightforward. We therefore consider a specific path in the neigh-

[1] Both exact and ordinal concepts have been considered in the literature. For a function P on the space of strategy profiles to be an *exact* potential (a *weighted* potential), the difference in a player's payoff resulting from a unilateral change of her strategy must precisely (up to a positive factor) equal the corresponding difference in P. For a function to be an *ordinal potential* (a *generalized ordinal potential)*, any weak or strict gain (any strict gain) in a player's payoff resulting from a unilateral change of its strategy must be reflected by a corresponding gain in P.

[2] Applications of potential methods are vast and include, for example, the analysis of oligopolistic markets (Slade, 1994), learning processes (Monderer and Shapley, 1996b; Fudenberg and Levine, 1998; Young, 2004), population dynamics (Sandholm, 2001, 2009; Cheung, 2014), the robustness of equilibria (Frankel et al., 2003; Morris and Ui, 2005; Okada and Tercieux, 2012), the decomposition of games (Candogan et al., 2011), imitation strategies (Duersch et al., 2012), dynamics (Candogan et al., 2013a, 2013b), equilibrium existence (Voorneveld, 1997; Kukushkin, 1994, 2011), solution concepts (Peleg et al., 1996; Tercieux and Voorneveld, 2010), games with monotone best-response selections (Huang, 2002; Dubey et al., 2006; Jensen, 2010), supermodular and zero-sum games (Brânzei et al., 2003), and even mechanism design (Jehiel et al., 2008).

[3] Since any ordinal potential game is, in particular, a generalized ordinal potential game, we also obtain necessary conditions for the existence of an ordinal potential.

borhood of a fixed strategy profile x_N^*, and shrink it to virtually infinitesimal size, as if using a pantograph.[4] This differential approach leads to simple necessary conditions for the local feasibility of a generalized ordinal potential. In particular, it constitutes the basis of our main result, a local analogue of the Monderer-Shapley condition for the existence of an exact (or weighted) potential in smooth games.[5]

The analysis starts by considering strict improvement cycles that involve two players only. In this case, the existence of a generalized ordinal potential is shown to imply that the product of the slopes of any two players' mutual local best-response functions (or, more generally, the product of the corresponding cross-derivatives) must be nonnegative at any strategy profile at which at least two first-order conditions hold in the interior. Thus, borrowing the terminology familiar from contributions such as Bulow et al. (1985), Amir (1996), Dubey et al. (2006), and Monaco and Sabarwal (2016), we obtain as our first main necessary condition that the game must exhibit pairwise strategic substitutes or complements at any such point.

Next, we consider strict improvement cycles that involve more than two players. In the simplest case, the path runs along the edges of a small rectangular box that contains a fixed strategy profile x_N^* at its center. To formalize the resulting condition on cross-derivatives, we introduce a modified Jacobian at x_N^* by replacing all diagonal entries of the Jacobian of the game by zero and by multiplying all entries above the diagonal with negative one. It turns out that the existence of a particular strict improvement cycle corresponds precisely to the property that the modified Jacobian is *semipositive* (Fiedler and Pták, 1966).[6] We transform these conditions in several steps, using a powerful recursive characterization of semipositivity due to Johnson et al. (1994), as well as the technique of flipping around individual strategy spaces (Vives, 1990; Amir, 1996). These steps lead us ultimately to the local analogue of the Monderer-Shapley condition.

[4] A *pantograph* is a mechanical drawing instrument that allows creating copies of a plan on a different scale.

[5] To caution the reader, we stress that, despite a similarity in terminology, the present analysis is *not* directly related to the use of local potentials in the analysis of informational robustness (Morris, 1999; Frankel et al., 2003; Morris and Ui, 2005; Okada and Tercieux, 2012).

[6] The relevant elements of the theory of semipositive matrices will be reviewed below.

Finally, we apply our necessary conditions to two selected classes of games that traditionally have been studied by economists. These are the lottery contest with heterogeneous valuations, and a differentiated Bertrand game with linear demand and quadratic costs. For these classes of games, our necessary conditions turn out to be also sufficient for the existence of a generalized ordinal potential.

Related literature. Complete characterizations of exact and generalized ordinal potential games are known for the class of finite games (Monderer and Shapley, 1996a; Milchtaich, 1996). For the class of strategic games, Monderer and Shapley (1996a) provided a complete characterization of exact potential games. Voorneveld and Norde (1997) have shown that a strategic game admits an ordinal potential if and only there are no weak improvement cycles and an order condition is satisfied.[7] Exact potential games admit a convenient characterization in the class of smooth games, i.e., in the class of games with interval strategy spaces and twice continuously differentiable payoff functions. Specifically, a smooth game admits an exact potential if and only if the Jacobian of that game, i.e., the matrix of cross-derivatives of players' payoff functions, is globally symmetric (Monderer and Shapley, 1996a). This characterization extends in a straightforward way to weighted potential games. However, no *differentiable* characterization has been available up to this point for the classes of ordinal potential or generalized ordinal potential games.[8]

The rest of the paper is structured as follows. Section 2 contains preliminaries. Section 3 discusses strict improvement cycles involving two players. The case of more than two involved players is dealt with in Section 4. Section 5 presents the local analogue of the Monderer-Shapley condition. Applications are provided in Section 6. Section 7 concludes. All the proofs have been relegated to an Appendix.

[7] For a rigorous statement of this important result, we refer the reader to Voorneveld and Norde (1997).

[8] Monderer and Shapley (1996a, p. 135) wrote: "Unlike (weighted) potential games, ordinal potential games are not easily characterized. We do not know of any useful characterization, analogous to the one given in (4.1), for differentiable ordinal potential games." Since then, the problem has apparently remained unaddressed. See, e.g., the recent surveys by Mallozzi (2013), González-Sánchez and Hernández-Lerma (2016), or Lã et al. (2016).

2. Preliminaries

Let Γ be a *(noncooperative) n-player game* Γ with set of players $N = \{1, ..., n\}$, strategy space X_i for each player $i \in N$, and payoff function $u_i : X_N = X_1 \times ... \times X_n \to \mathbb{R}$ for each $i \in N$. Given a strategy profile $x_N \equiv (x_1, ..., x_n) \in X_N$, we denote by $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ the profile composed of the strategies chosen by the opponents of player i, so that $x_{-i} \in X_{-i} = X_1 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_n$. Thus, $u_i(x_N) = u_i(x_i, x_{-i})$, etc. The game Γ will be called *smooth* (Vives, 1999) if each $X_i \subseteq \mathbb{R}$ is a non-degenerate compact interval and u_i is twice continuously differentiable on X_N, for any $i \in N$.[9]

Following Monderer and Shapley (1996a), a noncooperative game Γ is called an *ordinal potential game* [a *generalized ordinal potential game*] if there exists a function $P : X_N \to \mathbb{R}$ such that

$$u_i(x_i, x_{-i}) > u_i(\widehat{x}_i, x_{-i}) \quad \underset{\text{[implies]}}{\text{if and only if}} \quad P(x_i, x_{-i}) > P(\widehat{x}_i, x_{-i}). \tag{1}$$

for any $i \in N$, $x_i \in X_i$, $\widehat{x}_i \in X_i$, and $x_{-i} \in X_{-i}$. Note that no continuity nor differentiability assumptions are imposed on the (generalized) ordinal potential functions.[10]

Given an n-player game Γ, a *strict improvement cycle* of length L is a finite nondegenerate sequence of strategy profiles

$$... \to x_N^0 \to x_N^1 \to ... \to x_N^{L-1} \to ... \tag{2}$$

in X_N with the property that, for any $l = 0, ..., L-1$, there is a player $i = \iota(l) \in N$ such that $x_{-i}^{l+1} = x_{-i}^l$ and $u_i(x_i^{l+1}, x_{-i}^l) > u_i(x_i^l, x_{-i}^l)$, where x_N^L should be read as x_N^0.

We then have the following useful result.

Lemma 1 (Voorneveld, 1997). *A generalized ordinal potential game does not admit any strict improvement cycle.*

[9] Thus, we restrict attention to one-dimensional strategy spaces. The extension to multidimensional strategy spaces is discussed in the conclusion.

[10] This point is worth being mentioned because, as pointed out by Voorneveld (1997), a continuous ordinal potential game need not, in general, admit a continuous ordinal potential. See also Peleg et al. (1996).

3. Strict improvement cycles involving two players

We start the analysis by considering local strict improvement cycles that involve precisely two players. For this case, the approach outlined in the introduction leads to the following observation.

Proposition 1. *Suppose that the smooth n-player game* Γ *admits a generalized ordinal potential. Then, for any two players* $i, j \in N$ *with* $i \neq j$*, it holds that*

$$\left\{ x_i > 0,\ x_j > 0,\ \textit{and}\ \frac{\partial u_i(x_N)}{\partial x_i} = \frac{\partial u_j(x_N)}{\partial x_j} = 0 \right\} \Rightarrow \frac{\partial^2 u_i(x_N)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N)}{\partial x_i \partial x_j} \geq 0. \tag{3}$$

Thus, if a smooth n-player game admits a generalized ordinal potential, then at any strategy profile at which marginal payoffs vanish in the interior for two players, the product of the corresponding cross-derivatives is nonnegative. In particular, any smooth generalized ordinal potential game necessarily exhibits pairwise strategic substitutes or complements at any interior Cournot-Nash equilibrium.

Proposition 1 implies that games in which one player's best-response function is strictly increasing and another player's best response function is strictly decreasing is never a generalized ordinal potential game. An example is the mixed oligopoly model by Singh and Vives (1984), in which one firm chooses price, and the other firm chooses quantity. Another instance is quantity competition between a dominant firm and several fringe firms (Bulow et al., 1985). Likewise, asymmetric contests (Dixit, 1987) never admit a generalized ordinal potential. Many similar examples, taken from diverse areas such as law enforcement, business strategy, and citizen protests, can be found in Tombak (2006) and in Monaco and Sabarwal (2016).

In games with more than two players, condition (3) requires *pairwise* strategic complements or substitutes only. Therefore, unless sum-aggregative,[11] an ordinal potential game need not exhibit either strategic complements or strategic substitutes at x_N. Indeed, flipping around the natural order of the strategy space of one of three players, say, may certainly destroy the

[11] See, e.g., Corchón (1994).

property of strategic complements or strategic substitutes, but it does not affect the pairwise property, nor does it affect the property of being an ordinal potential game.

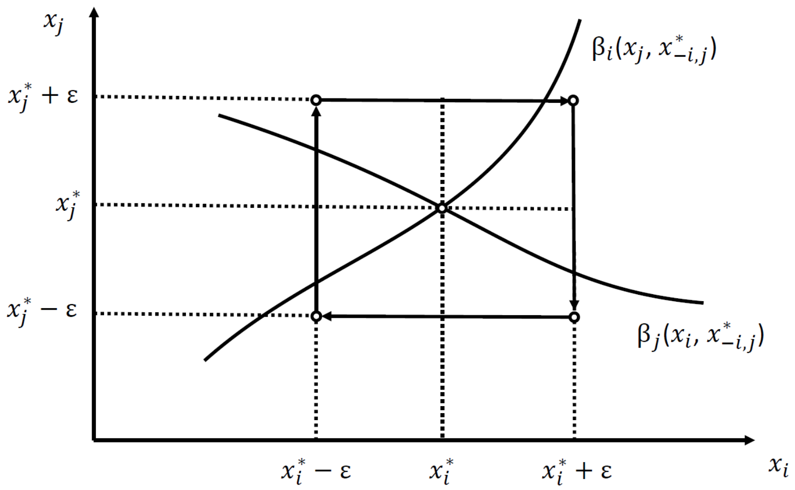

Figure 1. Constructing a strict improvement cycle involving two players.

To understand why Proposition 1 holds true, consider Figure 1. Keeping the strategy profile $x^*_{-i,j} = (x^*_1, ..., x^*_{i-1}, x^*_{i+1}, ..., x^*_{j-1}, x^*_{j+1}, ..., x^*_n)$ fixed, player i's best-response function[12] $\beta_i = \beta_i(x_j, x^*_{-i,j})$ is assumed to be well-defined and strictly *increasing* in player j's strategy x_j, while player j's best-response function $\beta_j = \beta_j(x_i, x^*_{-i,j})$ is strictly *decreasing* in player i's strategy x_i. Therefore,

$$\frac{\partial^2 u_i(x^*_N)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x^*_N)}{\partial x_i \partial x_j} < 0, \tag{5}$$

in conflict with the necessary condition. And indeed, for $\varepsilon > 0$ small enough, the circular path starting at the upper left corner of the depicted square and running clockwise around it,

[12] As usual, provided that $\partial^2 u_i / \partial x_i^2 < 0$ holds globally, player i's marginal condition defines a function $\beta_i = \beta_i(x_{-i})$ that maps any vector $x_{-i} \in X_{-i}$ to a unique $\beta_i(x_{-i}) \in X_i$ such that $\partial u_i(\beta_i(x_{-i}), x_{-i})/\partial x_i = 0$ holds in the interior. We will refer to β_i as player i's *best-response function.* In particular, for any other player $j \neq i$, we may refer to

$$\sigma_{ij}(x_N) = \frac{\partial \beta_i(x_{-i})}{\partial x_j} = -\frac{\partial^2 u_i(x_N)/\partial x_j \partial x_i}{\partial^2 u_i(x_N)/\partial x_i^2} \tag{4}$$

as the *slope* of player i's best-response function with respect to player j (in the interior). Thus, $\sigma_{ij}(x_N) \geq 0$ (≤ 0) if and only if Γ exhibits strategic complements (strategic substitutes) between i and j at x_N.

i.e., the finite sequence

$$\begin{aligned} &... \to (x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*) \to (x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*) \to \\ &\to (x_i^* + \varepsilon, x_j^* - \varepsilon, x_{-i,j}^*) \to (x_i^* - \varepsilon, x_j^* - \varepsilon, x_{-i,j}^*) \to ..., \end{aligned} \tag{6}$$

constitutes a strict improvement cycle, as will be explained now. To start with, consider the strategy change corresponding to the upper side of the square. Then, with ε small, player i's payoff is first increasing (over a longer section of the side) and then decreasing (over a shorter section of the side).

The point to note is that, as a consequence of smoothness of payoffs at the Cournot-Nash equilibrium, player i's payoff function along the upper side of the square may be approximated arbitrarily well by a parabola opening downwards, provided the square is small enough. As the parabola is symmetric around its peak, the payoff difference for player i, when switching from strategy $x_i - \varepsilon$ to $x_i + \varepsilon$, will be overall positive.[13] Similar considerations apply to the remaining three sides of the square. In fact, at the bottom side, there is no trade-off because player i's marginal payoff is always negative there. Thus, in sum, one may construct a strict improvement cycle that leads around the equilibrium. As noted in the previous section, however, this is incompatible with the existence of a generalized ordinal potential.

4. Strict improvement cycles involving more than two players

In this section, we consider strict improvement cycles that involve more than two players. We will initially restrict attention to a narrow class of paths in which the players involved consecutively raise their respective strategies, taking turns in their natural order, and subsequently lower their strategies, following the same order. Figure 2 illustrates a path of this kind for the case of three players.[14]

[13] There is a minor technical subtlety here in so far that the payoff difference approaches zero as ε goes to zero. However, as shown in the Appendix with the help of a careful limit consideration, the payoff difference approaches zero *from above* since the corresponding cross-derivative is positive. This turns out to be sufficient to settle the trade-off for a sufficiently small but still positive ε.

[14] In contrast to the previous section, we will allow for edges that are not necessarily of equal length.

It turns out that the local feasibility of a strict improvement cycle is, for general $n \geq 2$, determined by the *semi-positivity* (Fiedler and Pták, 1966; Johnson et al., 1994) of a modified matrix that is derived from the Jacobian of the game by changing the sign of all entries above the diagonal, and by replacing all entries on the diagonal by zero.

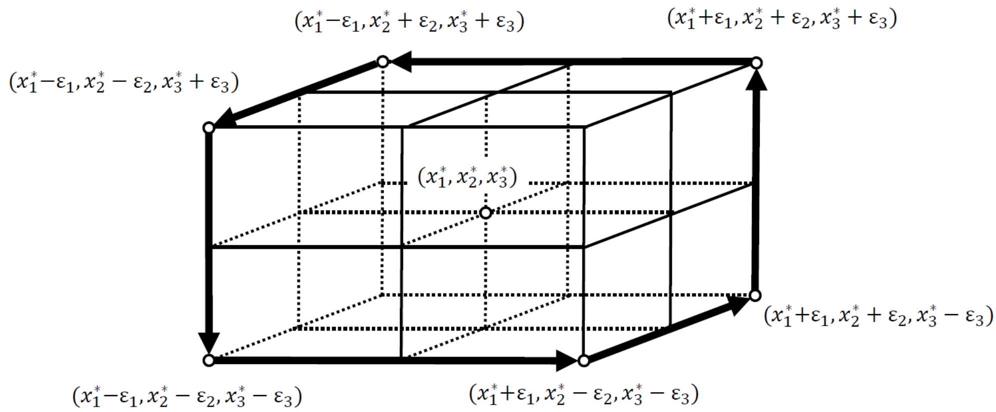

Figure 2. Constructing a strict improvement cycle involving three players.

Formally, let $\lambda_N = (\lambda_1, ..., \lambda_n)^T \in \mathbb{R}^n$ be a vector, where the T indicates transposition, as usual. We will write $\lambda_N > 0$ if all entries of λ_N are positive, i.e., if $\lambda_i > 0$ for all $i = 1, ..., n$.

Definition (Fiedler and Pták, 1966). *A square matrix $A \in \mathbb{R}^{n \times n}$ is **semipositive** if there exists $\lambda_N > 0$ such that $A\lambda_N > 0$.*[15]

An analysis of the conditions for the described path to constitute a strict improvement cycle leads to the following observation.

Proposition 2. *Suppose that the smooth n-player game Γ admits a generalized ordinal potential. Then, at any strategy profile $x_N \in X_N$ such that*

$$x_1 > 0, ..., x_n > 0, \text{ and } \frac{\partial u_1(x_N)}{\partial x_1} = ... = \frac{\partial u_n(x_N)}{\partial x_n} = 0, \tag{7}$$

[15]Semipositivity generalizes the concept of a P-matrix (Gale and Nikaidô, 1965). Intuitively, the fact that A is semipositive says that the interior of the convex cone generated by the columns of A intersects the positive orthant $\mathbb{R}^n_{++} = \{z_N \in \mathbb{R}^n : z_N > 0\}$.

the modified Jacobian

$$J(x_N) = \begin{pmatrix} 0 & -\dfrac{\partial^2 u_1(x_N)}{\partial x_2 \partial x_1} & \cdots & -\dfrac{\partial^2 u_1(x_N)}{\partial x_n \partial x_1} \\ \dfrac{\partial^2 u_2(x_N)}{\partial x_1 \partial x_2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\dfrac{\partial^2 u_{n-1}(x_N)}{\partial x_n \partial x_{n-1}} \\ \dfrac{\partial^2 u_n(x_N)}{\partial x_1 \partial x_n} & \cdots & \dfrac{\partial^2 u_n(x_N)}{\partial x_{n-1} \partial x_n} & 0 \end{pmatrix} \tag{8}$$

cannot be semipositive.

This result may be summarized as follow. Consider a smooth generalized ordinal potential game, and take any interior strategy profile x_N at which marginal payoffs vanish for all players. For instance, x_N could be an interior Cournot-Nash equilibrium. Then, the modified Jacobian at x_N must not be semipositive.

There are two basic ways to generate even more stringent necessary conditions. First, one may permute the order in which players change their respective strategies. Second, one may flip around the natural order of individual strategy spaces. The general treatment of these extensions, accomplished in the working paper, turns out to be obsolete for the purpose of the present analysis. However, we will use simple instances of permutations and flipped strategy spaces in the proof of our main result.[16]

5. A local analogue of the Monderer-Shapley condition

From the necessary conditions identified in the previous sections, one may derive the following theorem, which is the main result of the present paper.

Theorem 1. *Consider an interior strategy profile x_N^* in a smooth n-player game Γ such that*

$$\frac{\partial u_1(x_N^*)}{\partial x_1} = ... = \frac{\partial u_N(x_N^*)}{\partial x_N} = 0, \tag{9}$$

[16] It is probably also worthwhile mentioning that no additional conditions can be obtained by considering strategy profiles at which less than two first-order conditions hold, or by allowing for more complicated local paths. For rigorous proofs of these facts, the interested reader is likewise referred to the working paper version.

and such that all cross-derivatives $\{\partial u_i(x_N^*)/\partial x_j \partial x_i\}_{i,j \in N \text{ s.t. } i \neq j}$ *are nonzero. If* Γ *admits a generalized ordinal potential, then there exist positive weights* $w_1(x_N^*) > 0, ..., w_n(x_N^*) > 0$ *such that*

$$w_i(x_N^*) \frac{\partial u_i(x_N^*)}{\partial x_j \partial x_i} = w_j(x_N^*) \frac{\partial u_j(x_N^*)}{\partial x_i \partial x_j} \qquad (i, j \in N, j \neq i). \tag{10}$$

Thus, as outlined in the Introduction, the existence of an ordinal potential implies a local property that is analogous to the global differentiable condition for a weighted potential game.

To see the theorem at work, assume that payoffs admit the representation

$$u_i(x_N) = x_i \cdot \phi(x_N) \qquad (i \in N), \tag{11}$$

where $X_i \subseteq \mathbb{R}_{++}$ for all $i \in N$, and $\phi : X_N \rightarrow \mathbb{R}$ is an arbitrary twice continuously differentiable function that does not depend on i. Then, as pointed out by Kukushkin (1994), the mapping

$$P(x_N) = x_1 \cdot ... \cdot x_N \cdot \phi(x_N) \tag{12}$$

is an ordinal potential.[17] Suppose that x_N^* is an interior Cournot-Nash equilibrium. Then, Theorem 1 implies the existence of weights $w_1(x_N^*) > 0, ..., w_n(x_N^*) > 0$ such that (10) holds. This is indeed the case for

$$w_i(x_N^*) = \frac{1}{x_i^*}, \tag{13}$$

as can be checked in a straightforward way.[18]

Theorem 1 assumes that all cross-derivatives are non-vanishing. As the following example shows, this assumption cannot be dropped without losing the conclusion of Theorem 1.

[17] For a recent application, see Nocke and Schutz (2018).

[18] Making use of player j's first-order condition

$$x_j^* \frac{\partial \phi(x_N^*)}{\partial x_j} + \phi(x_N^*) = 0, \tag{14}$$

we see that

$$w_i(x_N^*) \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} = \frac{1}{x_i^*} \cdot \left\{ x_i^* \frac{\partial^2 \phi(x_N^*)}{\partial x_j \partial x_i} + \frac{\partial \phi(x_N^*)}{\partial x_j} \right\} = \frac{\partial^2 \phi(x_N^*)}{\partial x_j \partial x_i} - \frac{\phi(x_N^*)}{x_j^* x_i^*} \tag{15}$$

is symmetric with respect i and j. Thus, (10) indeed holds for the suggested local weights.

Example 1. Let $N = \{1, 2\}$, $X_1 = X_2 = [-1, 1]$, and

$$u_1(x_1, x_2) = -(x_1 + x_2)^2, \tag{16}$$

$$u_2(x_1, x_2) = -(x_1 + x_2)^6. \tag{17}$$

This game is easily seen to admit the ordinal potential $P(x_1, x_2) = u_1(x_1, x_2)$. However, at the interior Cournot-Nash equilibrium $x_N^* = (0, 0)$, the cross-derivatives are given by

$$\frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} = -2 \text{ and } \frac{\partial^2 u_2(x_N^*)}{\partial x_1 \partial x_2} = 0, \tag{18}$$

in conflict with relationship (10).

Thus, intuitively, the assumption that cross-derivatives do not vanish is needed because strictly monotone transformations of a player's payoff, under which the class of ordinal potential games is invariant, may possess points with zero slope.

The proof of Theorem 1 has three main steps. First, one considers a variety of strict improvement cycles involving either two or three players to show that, at any profile x_N^*, and for any set $\{i, j, k\} \subseteq N$ of pairwise different players,

$$\begin{gathered}\left\{x_i^* > 0,\, x_j^* > 0,\, x_k^* > 0, \text{ and } \frac{\partial u_i(x_N^*)}{\partial x_i} = \frac{\partial u_j(x_N^*)}{\partial x_j} = \frac{\partial u_k(x_N^*)}{\partial x_k} = 0\right\} \\ \Rightarrow \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_k \partial x_j} \cdot \frac{\partial^2 u_k(x_N^*)}{\partial x_i \partial x_k} = \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} \cdot \frac{\partial^2 u_k(x_N^*)}{\partial x_j \partial x_k} \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_k \partial x_i}.\end{gathered} \tag{19}$$

Then, an induction argument is used to obtain an analogous result for more than three players. In a final step, the weight for player i is defined as

$$w_i(x_N^*) = \left(\left|\frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1}\right| \cdot \ldots \cdot \left|\frac{\partial^2 u_{i-1}(x_N^*)}{\partial x_i \partial x_{i-1}}\right|\right) \cdot \left(\left|\frac{\partial^2 u_{i+1}(x_N^*)}{\partial x_i \partial x_{i+1}}\right| \cdot \ldots \cdot \left|\frac{\partial^2 u_n(x_N^*)}{\partial x_{n-1} \partial x_n}\right|\right), \tag{20}$$

and the consistency relations derived in the previous two steps are used to show that these weights have the predicted property (10).

6. Applications

In this section, we review two classes of games traditionally studied by economists, and observe that, for these games, our necessary conditions are also sufficient.[19]

6.1 Contests

In the n-player *lottery contest* with valuations $V_1 > 0, ..., V_n > 0$, player i's payoff is given by

$$u_i(x_1, ..., x_n) = \begin{cases} \dfrac{x_i}{x_1 + ... + x_n} V_i - x_i, & \text{if } x_1 + ... + x_n > 0 \\ \\ \dfrac{1}{n} & \text{if } x_1 + ... + x_n = 0, \end{cases} \tag{21}$$

where we assume that $X_1 = ... = X_n = [0, V^{\max}]$, where $V^{\max} = \max_{i \in N} V_i$. It follows from a general result of Szidarovszky and Okuguchi (1997) that this game has a unique (yet not necessarily interior) Cournot-Nash equilibrium $x_N^* = (x_1^*, ..., x_n^*)$.

Rather than applying our criterion to the n-player equilibrium, we will consider an equilibrium in the two-player game between arbitrary players $i, j \in N$ with $j \neq i$, assuming that all remaining players remain inactive. So let $x_N^\# = (x_i^\#, x_j^\#, x_{-i,j}^\#) \in X_N$ be such that the following conditions hold:

$$x_i^\# > 0,\ x_j^\# > 0,\ x_{-i,j}^\# = (0, ..., 0) \in \mathbb{R}^{n-2}, \text{ and} \tag{22}$$

$$\frac{\partial u_i(x_N^\#)}{\partial x_i} = \frac{\partial u_j(x_N^\#)}{\partial x_j} = 0. \tag{23}$$

In the bilateral game between players i and j, equilibrium efforts are given by the well-known expressions (cf. Konrad, 2009)

$$x_i^\# = \frac{V_i^2 V_j}{(V_i + V_j)^2} \text{ and } x_j^\# = \frac{V_i V_j^2}{(V_i + V_j)^2}. \tag{24}$$

Cross-derivatives are given by

$$\frac{\partial^2 u_i(x_N^\#)}{\partial x_j \partial x_i} = \frac{x_i^\# - x_j^\#}{(x_i^\# + x_j^\#)^3} V_i = \frac{V_i^2 - V_j^2}{(V_i V_j)^2}. \tag{25}$$

[19] Additional illustrations can be found in the working paper.

An analogous expression may be derived for player j. We therefore see that the necessary condition summarized in Proposition 1 holds if and only if $V_i = V_j$. On the other hand, if $V_1 = ... = V_n$, then the lottery contest can be shown to admit a generalized ordinal potential (Ewerhart and Kukushkin, 2018). Thus, in the case of the lottery contest with heterogeneous valuations, our criterion is indeed not only necessary but also sufficient.[20]

6.2 Differentiated Bertrand competition

There are $n \geq 2$ firms $i = 1, ..., n$ with differentiated products. Suppose that each firm $i \in N$ chooses a price $p_i \geq 0$, and subsequently sells a quantity

$$q_i(p_N) = Q_i - s_i p_i + \sum_{j \neq i} \theta_{ij} p_j, \tag{26}$$

where $Q_i > 0$, $s_i > 0$, and $\theta_{ij} \neq 0$ are arbitrary parameters.[21] Firm i's cost function is given by $C_i(q_i) = c_i q_i^2$, where $c_i > 0$. Thus, firm i's profit reads

$$u_i(p_N) = p_i q_i(p_N) - C_i(q_i(p_N)) \qquad (i \in N). \tag{27}$$

Suppose that there exists a price vector $p_N^* > 0$ at which marginal profits vanish (and at which profits are positive), for all firms.[22] Cross-derivatives are given by

$$\frac{\partial^2 u_i(p_N^*)}{\partial p_j \partial p_i} = \theta_{ij}(1 + 2 s_i c_i) \qquad (i, j \in N \text{ with } j \neq i) \tag{28}$$

Note that $1 + 2s_i c_i > 0$ for $i \in N$. Therefore, from Theorem 1, necessary for the existence of a generalized ordinal potential is that[23]

$$\{i, j \in N \text{ s.t. } i \neq j\} \Rightarrow \text{sgn}(\theta_{ij}) = \text{sgn}(\theta_{ji}), \tag{29}$$

[20] This example illustrates likewise the local nature of our conditions. Indeed, for $x_i \neq x_j$, the product of the cross-derivatives is negative in the lottery contest.

[21] If $\theta_{ij} = 0$ for some $i \neq j$, then the necessary and sufficient conditions (29-30) identified below must be complemented by additional conditions analogous to (30) for any number $m \in \{4, ..., n\}$ of players. Moreover, the functional form of the potential may differ somewhat from (31). In a nutshell, one first notes that, for a generalized ordinal potential to exist in the Bertrand game, it is necessary that, for any $i, j \in N$ with $i \neq j$, it holds that $\theta_{ij} = 0$ if and only if $\theta_{ji} = 0$. Then, in any connected component in the directed weighted graph defined by the bilateral price externalities $\{\theta_{ij}\}$, one chooses a spanning tree and defines a potential weight for a firm (or node) analogous to (33) by taking the product over all absolute price externalities represented as weights in the tree directed towards the firm. The details are omitted.

[22] This is the case, for instance, if $\max_{j \neq i} |\theta_{ij}|$ is sufficiently small.

[23] As usual, sgn(.) denotes the *sign function*, with $\text{sgn}(z) = 1$ if $z > 0$, $\text{sgn}(z) = 0$ if $z = 0$, and $\text{sgn}(z) = -1$ if $z < 0$.

and

$$\{i, j, k \in N \text{ s.t. } i \neq j \neq k \neq i\} \Rightarrow \theta_{ij}\theta_{jk}\theta_{ki} = \theta_{ji}\theta_{kj}\theta_{ik}. \tag{30}$$

Conversely, we have the following observation.

Proposition 3. *Suppose that conditions (29) and (30) hold. Then,*

$$P(p_N) = \sum_{i=1}^{n} \widehat{w}_i p_i \left\{ Q_i \left(1 + \frac{p_i}{2p_i^0}\right) + \sum_{j=1}^{i-1} \theta_{ij} p_j \right\}, \tag{31}$$

with

$$p_i^0 = \frac{(1 + 2s_i c_i)\, Q_i}{2s_i(1 + s_i c_i)} \qquad (i \in N), \text{ and} \tag{32}$$

$$\widehat{w}_i = (|\theta_{12}| \cdot \ldots \cdot |\theta_{i-1i}|) \cdot (|\theta_{i+1i}| \cdot \ldots \cdot |\theta_{nn-1}|) \qquad (i \in N), \tag{33}$$

is a generalized ordinal potential for the differentiated Bertrand game.

Thus, also in this class of games, our conditions turn out to both necessary and sufficient for the existence of a generalized ordinal potential.[24]

7. Concluding remarks

In this paper, we have identified conditions necessary for the existence of a generalized ordinal potential in any game with interval strategy spaces and twice continuously differentiable payoff functions. For selected classes of games, including lottery contests with heterogeneous valuations, and differentiated Bertrand games with linear demand and quadratic costs, these conditions are also sufficient. Thus, a first step towards the differentiable characterization of ordinal and generalized ordinal potential games has been made.

Further extensions are possible. For example, using similar methods, a necessary differentiable condition may be obtained from the analysis of non-local strict improvement cycles. This idea is developed in the working paper version. There as well, we extend Propositions 1

[24] In addition, the example shows that our necessary conditions may indeed be indicative of sufficient conditions.

and 2 to games with multidimensional strategy spaces. However, given the lack of a suitable generalization of the notion of semipositivity, it has to remain an open question if the local analogue of the Monderer-Shapley condition extends accordingly.[25]

Echenique (2004) has shown that, generically, a finite two-player ordinal potential game is a game of strategic complements (potentially after reordering strategic spaces), yet an ordinal potential game with more than two players need not generically be a game of strategic complements. While it is not straightforward to compare the two settings, some intuitions carry over. In particular, our findings provoke the question if any finite ordinal potential game with more than two players can be turned, by reordering of strategy spaces, into a game of pairwise strategic complements and substitutes. We do not know the answer to this question.

Any smooth exact potential game with negative definite Hessian is a concave game in the sense of Rosen (1965).[26] The converse, however, is not true in general. In fact, our results imply that a concave game need not even admit a generalized ordinal potential. Indeed, the lottery contest is concave (Ewerhart and Quartieri, 2016), but as seen above, it does not admit a generalized ordinal potential unless valuations are homogeneous.

There are three natural classes of smooth games that satisfy our most stringent necessary conditions. These are symmetric two-player zero-sum games, supermodular sum-aggregative game, and symmetric games that admit only symmetric equilibria.[27] To the extent that necessary conditions are indicative of sufficient conditions, one might speculate that these classes of games, potentially under additional technical conditions, have the property that an ordinal potential can be constructed locally.[28]

[25] A multi-dimensional variant of the Monderer-Shapley condition for exact potential games can be found in Deb (2008, 2009).

[26] See Neyman (1997), Ui (2008), and Hofbauer and Sandholm (2009).

[27] Smooth symmetric games admit at least one symmetric equilibrium under standard assumptions (Moulin, 1986, p. 115). However, there are also large classes of economically relevant symmetric games that admit only asymmetric pure-strategy Nash equilibria (cf. Amir et al., 2010).

[28] The author is presently exploring the validity of these conjectures.

Appendix: Proofs

Proof of Proposition 1. By contradiction. Suppose that, at some profile $x_N \in X_N$, and for some players i and j with $i \neq j$, we have

$$x_i > 0,\ x_j > 0,\ \frac{\partial u_i(x_N^*)}{\partial x_i} = \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \text{ and} \tag{34}$$

$$\frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} < 0. \tag{35}$$

By renaming players, if necessary, we may assume w.l.o.g. that

$$\frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} > 0 > \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j}, \tag{36}$$

which corresponds to the case shown in Figure 1. It is claimed now that, for any sufficiently small $\varepsilon > 0$, the payoff difference corresponding to the upper side of the square satisfies

$$\Delta_i^+(\varepsilon) \equiv u_i(x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*) - u_i(x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*) > 0. \tag{37}$$

To prove this, we determine the second-order Taylor approximation of $\Delta_i^+(\varepsilon)$ at $\varepsilon = 0$. Writing $f(\varepsilon)$ for $\Delta_i^+(\varepsilon)$, our differentiability assumptions combined with Taylor's theorem imply that there is a remainder term $r(.)$ with $\lim_{\varepsilon \to 0} r(\varepsilon) = 0$ such that for any sufficiently small $\varepsilon > 0$,

$$f(\varepsilon) = f(0) + f'(0)\varepsilon + \frac{1}{2} f''(0)\varepsilon^2 + r(\varepsilon)\varepsilon^2. \tag{38}$$

Clearly, $f(0) = 0$. As for the first derivative $f'(0)$, one obtains

$$\begin{aligned}\frac{\partial \Delta_i^+(\varepsilon)}{\partial \varepsilon} &= \left\{ \frac{\partial u_i(x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*)}{\partial x_i} + \frac{\partial u_i(x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*)}{\partial x_j} \right\} \\ &\quad - \left\{ -\frac{\partial u_i(x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*)}{\partial x_i} + \frac{\partial u_i(x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*)}{\partial x_j} \right\}.\end{aligned} \tag{39}$$

Evaluating at $\varepsilon = 0$, and subsequently exploiting the necessary first-order condition for player i at the interior equilibrium x_N^*, we find

$$f'(0) = \frac{\partial \Delta_i^+(0)}{\partial \varepsilon} = 2 \cdot \frac{\partial u_i(x_N^*)}{\partial x_i} = 0. \tag{40}$$

Next, consider the second derivative of $\Delta_i^+(\varepsilon)$ at $\varepsilon = 0$, i.e.,

$$\frac{\partial^2 \Delta_i^+(0)}{\partial \varepsilon^2} = \left\{ \frac{\partial^2 u_i(x_N^*)}{\partial x_i^2} + \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} + \frac{\partial^2 u_i(x_N^*)}{\partial x_i \partial x_j} + \frac{\partial^2 u_i(x_N^*)}{\partial x_j^2} \right\} \tag{41}$$

$$- \left\{ \frac{\partial^2 u_i(x_N^*)}{\partial x_i^2} - \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} - \frac{\partial^2 u_i(x_N^*)}{\partial x_i \partial x_j} + \frac{\partial^2 u_i(x_N^*)}{\partial x_j^2} \right\}$$

$$= 2 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} + 2 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_i \partial x_j}. \tag{42}$$

Invoking Schwarz's theorem regarding the equality of cross-derivatives for twice continuously differentiable functions, and subsequently using (36), one obtains

$$f''(0) = \frac{\partial^2 \Delta_i^+(0)}{\partial \varepsilon^2} = 4 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} > 0. \tag{43}$$

In sum, we have shown that $f'(0) = f(0) = 0$ and $f''(0) > 0$. Thus, using (38), it follows that $\Delta_i^+(\varepsilon) > 0$ for any sufficiently small $\varepsilon > 0$. Analogous arguments can be used to deal with the other three sides of the square. Specifically, one defines

$$\Delta_j^+(\varepsilon) = u_j(x_j^* - \varepsilon, x_i^* + \varepsilon, x_{-i,j}^*) - u_j(x_j^* + \varepsilon, x_i^* + \varepsilon, x_{-i,j}^*), \tag{44}$$

$$\Delta_i^-(\varepsilon) = u_i(x_i^* - \varepsilon, x_j^* - \varepsilon, x_{-i,j}^*) - u_i(x_i^* + \varepsilon, x_j^* - \varepsilon, x_{-i,j}^*), \tag{45}$$

$$\Delta_j^-(\varepsilon) = u_j(x_j^* + \varepsilon, x_i^* - \varepsilon, x_{-i,j}^*) - u_j(x_j^* - \varepsilon, x_i^* - \varepsilon, x_{-i,j}^*), \tag{46}$$

and now readily verifies that

$$\frac{\partial \Delta_j^+(0)}{\partial \varepsilon} = (-2) \cdot \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \tag{47}$$

$$\frac{\partial \Delta_i^-(0)}{\partial \varepsilon} = (-2) \cdot \frac{\partial u_i(x_N^*)}{\partial x_i} = 0, \tag{48}$$

$$\frac{\partial \Delta_j^-(0)}{\partial \varepsilon} = 2 \cdot \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \tag{49}$$

and that

$$\frac{\partial^2 \Delta_j^+(0)}{\partial \varepsilon^2} = (-4) \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} > 0, \tag{50}$$

$$\frac{\partial^2 \Delta_i^-(0)}{\partial \varepsilon^2} = 4 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} > 0, \tag{51}$$

$$\frac{\partial^2 \Delta_j^-(0)}{\partial \varepsilon^2} = (-4) \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} > 0. \tag{52}$$

It follows that $\Delta_i^+(\varepsilon) > 0$, $\Delta_j^+(\varepsilon) > 0$, $\Delta_i^-(\varepsilon) > 0$, and $\Delta_j^-(\varepsilon) > 0$ all hold for $\varepsilon > 0$ small enough. But then, the finite sequence (6) is a strict improvement cycle, which is incompatible with the existence of a generalized ordinal potential by Lemma 1. $\square$

Proof of Proposition 2. Suppose that the modified Jacobian $J = J(x_N)$ is semipositive. Then, by definition, there exists a vector $\lambda_N = (\lambda_1, ..., \lambda_n)^T \in \mathbb{R}^n$ with $\lambda_N > 0$ such that $J\lambda_N > 0$. Consider the finite sequence

$$\begin{aligned}
... \to x_N^{(1,+)}(\varepsilon) &= (x_1^* + \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, ..., x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \to \\
\to x_N^{(2,+)}(\varepsilon) &= (x_1^* + \lambda_1\varepsilon, x_2^* + \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, ..., x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \to \\
&\vdots \\
\to x_N^{(n,+)}(\varepsilon) &= (x_1^* + \lambda_1\varepsilon, x_2^* + \lambda_2\varepsilon, x_3^* + \lambda_3\varepsilon, ..., x_{n-1}^* + \lambda_{n-1}\varepsilon, x_n^* + \lambda_n\varepsilon) \to \\
\to x_N^{(1,-)}(\varepsilon) &= (x_1^* - \lambda_1\varepsilon, x_2^* + \lambda_2\varepsilon, x_3^* + \lambda_3\varepsilon, ..., x_{n-1}^* + \lambda_{n-1}\varepsilon, x_n^* + \lambda_n\varepsilon) \to \\
\to x_N^{(2,-)}(\varepsilon) &= (x_1^* - \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* + \lambda_3\varepsilon, ..., x_{n-1}^* + \lambda_{n-1}\varepsilon, x_n^* + \lambda_n\varepsilon) \to \\
&\vdots \\
\to x_N^{(n,-)}(\varepsilon) &= (x_1^* - \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, ..., x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \to ...,
\end{aligned} \tag{53}$$

where $\varepsilon > 0$ is a small constant as before. Figure 2 illustrates this path for $n = 3$, where the rectangular-shaped box has sides of respective length $\varepsilon_i = \lambda_i\varepsilon$ for $i = 1, 2, 3$. It is claimed that, for any $\varepsilon > 0$ sufficiently small, the following four conditions hold:

(i) player 1's payoff at $x_N^{(1,+)}(\varepsilon)$ is strictly higher than at $x_N^{(n,-)}(\varepsilon)$;

(ii) for $i = 2, ..., n$, player i's payoff at $x_N^{(i,+)}(\varepsilon)$ is strictly higher than at $x_N^{(i-1,+)}(\varepsilon)$;

(iii) player 1's payoff at $x_N^{(1,-)}(\varepsilon)$ is strictly higher than at $x_N^{(n,+)}(\varepsilon)$;

(iv) for $i = 2, ..., n$, player i's payoff at $x_N^{(i,-)}(\varepsilon)$ is strictly higher than at $x_N^{(i-1,-)}(\varepsilon)$.

To establish (i), we proceed as in the proof of Proposition 1, and consider the first two derivatives of the payoff difference

$$\Delta^{(1,+)}(\varepsilon) = u_1(x_N^{(1,+)}(\varepsilon)) - u_1(x_N^{(n,-)}(\varepsilon)) \tag{54}$$

$$= u_1(x_1^* + \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, ..., x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \tag{55}$$

$$- u_1(x_1^* - \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, ..., x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon)$$

at $\varepsilon = 0$. The first derivative of $\Delta^{(1,+)}(\varepsilon)$ at $\varepsilon = 0$ is given by

$$\frac{\partial\Delta^{(1,+)}(0)}{\partial\varepsilon} = \left(\lambda_1\frac{\partial u_1(x_N^*)}{\partial x_1} - \lambda_2\frac{\partial u_1(x_N^*)}{\partial x_2} - ... - \lambda_n\frac{\partial u_1(x_N^*)}{\partial x_n}\right) \tag{56}$$

$$- \left(-\lambda_1\frac{\partial u_1(x_N^*)}{\partial x_1} - \lambda_2\frac{\partial u_1(x_N^*)}{\partial x_2} - ... - \lambda_n\frac{\partial u_1(x_N^*)}{\partial x_n}\right)$$

$$= 2\lambda_1\frac{\partial u_1(x_N^*)}{\partial x_1}. \tag{57}$$

Hence, from player 1's first-order condition,

$$\frac{\partial\Delta^{(1,+)}(0)}{\partial\varepsilon} = 0. \tag{58}$$

Next, one considers the second derivative of $\Delta^{(1,+)}(\varepsilon)$ at $\varepsilon = 0$, i.e.,

$$\begin{aligned}\frac{\partial^2\Delta^{(1,+)}(0)}{\partial\varepsilon^2} = &\left\{(\lambda_1)^2\frac{\partial^2 u_1(x_N^*)}{\partial x_1^2} - \lambda_2\lambda_1\frac{\partial^2 u_1(x_N^*)}{\partial x_2\partial x_1} - ... - \lambda_n\lambda_1\frac{\partial^2 u_1(x_N^*)}{\partial x_n\partial x_1}\right.\\ &- \lambda_1\lambda_2\frac{\partial^2 u_1(x_N^*)}{\partial x_1\partial x_2} + (\lambda_2)^2\frac{\partial^2 u_1(x_N^*)}{\partial x_2^2} + ... + \lambda_n\lambda_2\frac{\partial^2 u_1(x_N^*)}{\partial x_n\partial x_2}\\ &\vdots\\ &\left.-\lambda_1\lambda_n\frac{\partial^2 u_1(x_N^*)}{\partial x_1\partial x_n} + \lambda_2\lambda_n\frac{\partial^2 u_1(x_N^*)}{\partial x_2\partial x_n} + ... + (\lambda_n)^2\frac{\partial^2 u_1(x_N^*)}{\partial x_n^2}\right\}\\ &- \left\{(\lambda_1)^2\frac{\partial^2 u_1(x_N^*)}{\partial x_1^2} + \lambda_2\lambda_1\frac{\partial^2 u_1(x_N^*)}{\partial x_2\partial x_1} + ... + \lambda_n\lambda_1\frac{\partial^2 u_1(x_N^*)}{\partial x_n\partial x_1}\right.\\ &+ \lambda_1\lambda_2\frac{\partial^2 u_1(x_N^*)}{\partial x_1\partial x_2} + (\lambda_2^2)\frac{\partial^2 u_1(x_N^*)}{\partial x_2^2} + ... + \lambda_n\lambda_2\frac{\partial^2 u_1(x_N^*)}{\partial x_n\partial x_2}\\ &\vdots\\ &\left.+ \lambda_1\lambda_n\frac{\partial^2 u_1(x_N^*)}{\partial x_1\partial x_n} + \lambda_2\lambda_n\frac{\partial^2 u_1(x_N^*)}{\partial x_2\partial x_n} + ... + (\lambda_n)^2\frac{\partial^2 u_1(x_N^*)}{\partial x_n^2}\right\}.\end{aligned} \tag{59}$$

Collecting terms, one obtains

$$\frac{\partial^2\Delta^{(1,+)}(0)}{\partial\varepsilon^2} = 2\lambda_1\left\{-\lambda_2\frac{\partial^2 u_1(x_N^*)}{\partial x_2\partial x_1} - \lambda_3\frac{\partial^2 u_1(x_N^*)}{\partial x_3\partial x_1} - ... - \lambda_n\frac{\partial^2 u_1(x_N^*)}{\partial x_n\partial x_1}\right\}. \tag{60}$$

Thus, using $\lambda_1 > 0$, and noting that the expression in the curly brackets corresponds to the first component of the vector $J\lambda_N > 0$, one arrives at

$$\frac{\partial^2 \Delta^{(1,+)}(0)}{\partial \varepsilon^2} > 0. \tag{61}$$

It follows that $\Delta^{(1,+)}(\varepsilon) > 0$ for any $\varepsilon > 0$ sufficiently small, which proves (i). To verify claims (ii) through (iv), define payoff differences

$$\Delta^{(i,+)}(\varepsilon) = u_i(x_N^{(i,+)}(\varepsilon)) - u_i(x_N^{(i-1,+)}(\varepsilon)) \qquad (i = 2, ..., n), \tag{62}$$

$$\Delta^{(1,-)}(\varepsilon) = u_1(x_N^{(1,-)}(\varepsilon)) - u_1(x_N^{(n,+)}(\varepsilon)), \tag{63}$$

$$\Delta^{(i,-)}(\varepsilon) = u_i(x_N^{(i,-)}(\varepsilon)) - u_i(x_N^{(i-1,+)}(\varepsilon)) \qquad (i = 2, ..., n). \tag{64}$$

Using players' necessary first-order conditions, it is straightforward to validate that

$$\frac{\partial \Delta^{(i,+)}(0)}{\partial \varepsilon} = 0 \qquad (i = 2, ..., n), \tag{65}$$

$$\frac{\partial \Delta^{(1,-)}(0)}{\partial \varepsilon} = 0, \tag{66}$$

$$\frac{\partial \Delta^{(i,-)}(0)}{\partial \varepsilon} = 0 \qquad (i = 2, ..., n). \tag{67}$$

Moreover, for $i = 2, ..., n$, calculations analogous to (59-60) yield

$$\frac{\partial^2 \Delta^{(i,+)}(0)}{\partial \varepsilon^2} = 2\lambda_i \left\{ \lambda_1 \frac{\partial^2 u_i(x_N^*)}{\partial x_1 \partial x_i} + ... + \lambda_{i-1} \frac{\partial^2 u_i(x_N^*)}{\partial x_{i-1} \partial x_i} \right. \tag{68}$$

$$\left. - \lambda_{i+1} \frac{\partial^2 u_i(x_N^*)}{\partial x_{i+1} \partial x_i} - ... - \lambda_n \frac{\partial^2 u_i(x_N^*)}{\partial x_n \partial x_i} \right\}$$

$$> 0, \tag{69}$$

where the expression in the curly brackets corresponds to the i's component of the vector $J\lambda_N > 0$. Finally, one notes that, from $d(-\varepsilon)^2 = d\varepsilon^2$, it follows that

$$\frac{\partial^2 \Delta^{(i,-)}(0)}{\partial \varepsilon^2} = \frac{\partial^2 \Delta^{(i,+)}(0)}{\partial \varepsilon^2} \qquad (i = 1, ..., n). \tag{70}$$

In sum, this proves (ii) through (iv). Thus, there exists a strict improvement cycle in the generalized ordinal potential game Γ. Since this is impossible, the proposition follows. $\square$

The three lemmas below will be used in the proof of Theorem 1. Following the literature, we will call a square matrix $A \in \mathbb{R}^{n \times n}$ *inverse nonnegative* if the matrix inverse A^{-1} exists and if, in addition, all entries of A^{-1} are nonnegative. The following lemma provides a useful recursive characterization of semipositivity.

Lemma A.1 (Johnson et al., 1994). *A square matrix $A \in \mathbb{R}^{n \times n}$ is semipositive if and only if at least one of the following two conditions holds:*

(i) A is inverse nonnegative;

(ii) there exists $m \in \{1, ..., n-1\}$ and a submatrix $\widehat{A} \in \mathbb{R}^{n \times m}$ obtained from A via deletion of $n - m$ columns, such that all $m \times m$ submatrices of $\widehat{A}$ are semipositive.

Using the lemma above, we may derive the following implication of Proposition 2.

Lemma A.2. *Suppose that the smooth n-player game Γ admits a generalized ordinal potential. Then, at any profile x_N^*, and for any set $\{i, j, k\} \subseteq N$ of pairwise different players,*

$$\left\{ x_i^* > 0,\, x_j^* > 0,\, x_k^* > 0, \text{ and } \frac{\partial u_i(x_N^*)}{\partial x_i} = \frac{\partial u_j(x_N^*)}{\partial x_j} = \frac{\partial u_k(x_N^*)}{\partial x_k} = 0 \right\} \tag{71}$$
$$\Rightarrow \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_k \partial x_j} \cdot \frac{\partial^2 u_k(x_N^*)}{\partial x_i \partial x_k} = \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} \cdot \frac{\partial^2 u_k(x_N^*)}{\partial x_j \partial x_k} \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_k \partial x_i}.$$

Proof. Fix some profile x_N^* and a set $\{i, j, k\} \subseteq N$ of pairwise different players such that

$$x_i^* > 0, x_j^* > 0, x_k^* > 0, \text{ and} \tag{72}$$
$$\frac{\partial u_i(x_N^*)}{\partial x_i} = \frac{\partial u_j(x_N^*)}{\partial x_j} = \frac{\partial u_k(x_N^*)}{\partial x_k} = 0. \tag{73}$$

Using the notation

$$\chi_{ij} = \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i}, \chi_{jk} = \frac{\partial^2 u_j(x_N^*)}{\partial x_k \partial x_j}, ..., \tag{74}$$

for cross-derivatives, it needs to be shown that

$$\chi_{ij} \cdot \chi_{jk} \cdot \chi_{ki} = \chi_{ji} \cdot \chi_{kj} \cdot \chi_{ik}. \tag{75}$$

Suppose first that Γ exhibits weak strategic complements at x_N^*, i.e.,

$$\chi_{ij} \geq 0,\ \chi_{ji} \geq 0,\ \chi_{ik} \geq 0,\ \chi_{ki} \geq 0,\ \chi_{jk} \geq 0, \text{ and } \chi_{kj} \geq 0. \tag{76}$$

Consider now a small circular path along the edges of a small three-dimensional rectangular-shaped box around x_N^*. Along the path, players i, j, and k move in this order, with i and k initially increasing their strategies, while j initially decreases her strategy. Since this corresponds to flipping around player j's strategy space, all cross-derivatives involving player j change sign, so that the corresponding modified Jacobian reads

$$J_3 = \begin{pmatrix} 0 & \chi_{ij} & -\chi_{ik} \\ -\chi_{ji} & 0 & \chi_{jk} \\ \chi_{ki} & -\chi_{kj} & 0 \end{pmatrix}. \tag{77}$$

By Proposition 2, J_3 cannot be inverse nonnegative. To prove (75), it suffices to show that the determinant of J_3,

$$|J_3| = \chi_{ij}\chi_{jk}\chi_{ki} - \chi_{ji}\chi_{kj}\chi_{ik}, \tag{78}$$

vanishes. To provoke a contradiction, suppose first that $|J_3| > 0$. Then, from weak strategic complements at x_N^*, all the entries of the matrix inverse of J_3,

$$(J_3)^{-1} = \frac{1}{|J_3|} \begin{pmatrix} \chi_{jk}\chi_{kj} & \chi_{ik}\chi_{kj} & \chi_{ij}\chi_{jk} \\ \chi_{ki}\chi_{jk} & \chi_{ik}\chi_{ki} & \chi_{ji}\chi_{ik} \\ \chi_{ji}\chi_{kj} & \chi_{ij}\chi_{ki} & \chi_{ij}\chi_{ji} \end{pmatrix}, \tag{79}$$

are nonnegative, in contradiction to the fact that J_3 is not inverse nonnegative. Hence, $|J_3| \leq 0$. Suppose next that $|J_3| < 0$. Then, by running through the above-considered path in the opposite direction (i.e., by exchanging the roles of players i and k), Proposition 2 implies that

$$\widehat{J}_3 = \begin{pmatrix} 0 & \chi_{kj} & -\chi_{ki} \\ -\chi_{jk} & 0 & \chi_{ji} \\ \chi_{ik} & -\chi_{ij} & 0 \end{pmatrix} \tag{80}$$

is not inverse nonnegative. Clearly, the determinant of $\widehat{J}_3$ is

$$\left|\widehat{J}_3\right| = \chi_{kj}\chi_{ji}\chi_{ik} - \chi_{jk}\chi_{ij}\chi_{ki} = -\left|J_3\right| > 0. \tag{81}$$

Hence, using an expression for the matrix inverse analogous to (79), all entries of $(\widehat{J}_3)^{-1}$ are seen to be nonnegative, in contradiction to the fact that $\widehat{J}_3$ is not inverse nonnegative. It follows that $|J_3| = 0$, which proves the claim in the case where Γ exhibits weak strategic complements at x_N^*. The case of weak strategic substitutes, where

$$\chi_{ij} \leq 0,\ \chi_{ji} \leq 0,\ \chi_{ik} \leq 0,\ \chi_{ki} \leq 0,\ \chi_{jk} \leq 0, \text{ and } \chi_{kj} \leq 0, \tag{82}$$

is entirely analogous, and therefore omitted. We are now in the position to address the general case. From Proposition 1, we know that Γ exhibits pairwise weak strategic complements or substitutes at x_N^*. Hence, up to a renaming of the players, there are only two remaining cases:

(i) Weak strategic complements at x_N^* between player i and each of players j and k, as well as strategic substitutes at x_N^* between players j and k;

(ii) Weak strategic substitutes at x_N^* between player i and each of players j and k, as well as weak strategic complements at x_N^* between players j and k.

In either case, by flipping around the strategy space of player i, the game may be transformed into a game that exhibits either weak strategic substitutes at x_N^* or weak strategic complements at x_N^*. Since the operation of flipping around individual strategy spaces does not affect the validity of equation (75), we find that the conclusion indeed holds in the general case. $\square$

Lemma A.3. *Suppose that the smooth n-player game Γ admits a generalized ordinal potential. Then, at any interior strategy profile x_N^* at which all first-order conditions hold, and for any set of pairwise distinct players $\{i_1, ..., i_m\} \subseteq N$ with $m \geq 3$, using the notation introduced in (74), it holds that*

$$\chi_{i_1 i_2} \cdot \chi_{i_2 i_3} \cdot ... \cdot \chi_{i_{m-1} i_m} \cdot \chi_{i_m i_1} = \chi_{i_2 i_1} \cdot \chi_{i_3 i_2} \cdot ... \cdot \chi_{i_m i_{m-1}} \cdot \chi_{i_1 i_m}, \tag{83}$$

provided that $\chi_{i_1 i_3} \cdot \chi_{i_3 i_1} \neq 0, ..., \chi_{i_1 i_{m-1}} \cdot \chi_{i_{m-i} i_1} \neq 0$.

Proof. The proof proceeds by induction. The case $m = 3$ follows directly from Lemma A.2. Suppose that $m \geq 4$, and let $\{i_1, i_2, ..., i_m\}$ be an arbitrary set of pairwise different players.

Suppose that the claim has been shown for any $m' \in \{3, 4, ..., m-1\}$. Then, a consideration of the two subsets $\{i_1, i_2, ..., i_{m-1}\}$ and $\{i_{m-1}, i_m, i_1\}$ shows that

$$\chi_{i_1 i_2} \cdot ... \cdot \chi_{i_{m-2} i_{m-1}} \cdot \chi_{i_{m-1} i_1} = \chi_{i_2 i_1} \cdot ... \cdot \chi_{i_{m-1} i_{m-2}} \cdot \chi_{i_1 i_{m-1}}, \text{ and} \tag{84}$$

$$\chi_{i_{m-1} i_m} \cdot \chi_{i_m i_1} \cdot \chi_{i_1 i_{m-1}} = \chi_{i_m i_{m-1}} \cdot \chi_{i_1 i_m} \cdot \chi_{i_{m-1} i_1}. \tag{85}$$

Combining the two equations via multiplication yields

$$\begin{aligned} & \left(\chi_{i_1 i_2} \cdot ... \cdot \chi_{i_{m-2} i_{m-1}} \cdot \chi_{i_{m-1} i_1}\right) \cdot \left(\chi_{i_{m-1} i_m} \cdot \chi_{i_m i_1} \cdot \chi_{i_1 i_{m-1}}\right) \\ & = \left(\chi_{i_2 i_1} \cdot ... \cdot \chi_{i_{m-1} i_{m-2}} \cdot \chi_{i_1 i_{m-1}}\right) \cdot \left(\chi_{i_m i_{m-1}} \cdot \chi_{i_1 i_m} \cdot \chi_{i_{m-1} i_1}\right). \end{aligned} \tag{86}$$

By assumption, $\chi_{i_1 i_{m-1}} \cdot \chi_{i_{m-1} i_1} \neq 0$. Hence, eliminating these common nonzero factors, (86) implies

$$\chi_{i_1 i_2} \cdot ... \cdot \chi_{i_{m-1} i_m} \cdot \chi_{i_m i_1} = \chi_{i_2 i_1} \cdot ... \cdot \chi_{i_m i_{m-1}} \cdot \chi_{i_1 i_m}, \tag{87}$$

as claimed. This concludes the induction step, and therefore proves the lemma. □

Proof of Theorem 1. Let x_N^* be an interior strategy profile such that all first-order conditions hold at x_N^* and such that $\chi_{ij} \neq 0$ for all $i \neq j$. We need to find positive constants $w_1 > 0, ..., w_n > 0$ such that

$$\chi_{ij} w_i = \chi_{ji} w_j \qquad (i, j \in N, j \neq i). \tag{88}$$

It is claimed that

$$w_i = (|\chi_{12}| \cdot ... \cdot |\chi_{i-1i}|) \cdot (|\chi_{i+1i}| \cdot ... \cdot |\chi_{nn-1}|) \qquad (i \in N) \tag{89}$$

does the job, with the usual convention that any empty product equals one. Clearly, it suffices to check (88) for $i < j$, because if $i > j$, one may just exchange the two sides of equation (88). Suppose first that $n = 2$. Then, (89) implies $w_1 = |\chi_{21}|$ and $w_2 = |\chi_{12}|$, and the claim follows directly from Proposition 1. Suppose next that $n \geq 3$. Splitting the product in the second bracket of (89), one obtains

$$\chi_{ij} w_i = (|\chi_{12}| \cdot ... \cdot |\chi_{i-1i}|) \cdot \mathrm{sgn}(\chi_{ij}) \cdot (|\chi_{i+1i}| \cdot ... \cdot |\chi_{jj-1}| \cdot |\chi_{ij}|) \cdot (|\chi_{j+1j}| \cdot ... \cdot |\chi_{nn-1}|)\,. \tag{90}$$

From Proposition 1,

$$\mathrm{sgn}(\chi_{ij}) = \mathrm{sgn}(\chi_{ji}). \tag{91}$$

Moreover, from Lemma A.3,

$$\chi_{i+1i} \cdot ... \cdot \chi_{jj-1} \cdot \chi_{ij} = \chi_{ii+1} \cdot ... \cdot \chi_{j-1j} \cdot \chi_{ji}. \tag{92}$$

Hence, using (91) and (92) in relationship (90) delivers

$$\chi_{ij} w_i = (|\chi_{12}| \cdot ... \cdot |\chi_{i-1i}|) \tag{93}$$

$$\cdot\, \mathrm{sgn}(\chi_{ji}) \cdot (|\chi_{ii+1}| \cdot ... \cdot |\chi_{j-1j}| \cdot |\chi_{ji}|) \tag{94}$$

$$\cdot\, (|\chi_{j+1j}| \cdot ... \cdot |\chi_{nn-1}|) \tag{95}$$

$$= \chi_{ji} \cdot (|\chi_{12}| \cdot ... \cdot |\chi_{j-1j}|) \cdot (|\chi_{j+1j}| \cdot ... \cdot |\chi_{nn-1}|) \tag{96}$$

$$= \chi_{ji} w_j. \tag{97}$$

This proves the claim and, hence, the theorem. □

Proof of Proposition 3. It will be shown that P is a weighted potential. Since any weighted potential is, in particular, a generalized ordinal potential, this is sufficient to prove the proposition. So consider some player $i \in N$, prices $p_i' \geq 0$ and $p_i'' \geq 0$, as well as a price vector $p_{-i} \in \mathbb{R}_+^{n-1}$. It is claimed that

$$P(p_i', p_{-i}) - P(p_i'', p_{-i}) = \frac{\widehat{w}_i}{1 + 2s_i c_i} \cdot (u_i(p_i', p_{-i}) - u_i(p_i'', p_{-i}))\,. \tag{98}$$

To see this, note first that

$$u_i(p_i, p_{-i}) = \left(p_i - c_i\left(Q_i - s_ip_i + \sum_{j\neq i}\theta_{ij}p_j\right)\right)\left(Q_i - s_ip_i + \sum_{j\neq i}\theta_{ij}p_j\right) \tag{99}$$

$$= (1+2s_ic_i)Q_ip_i - (1+s_ic_i)s_ip_i^2 + (1+2s_ic_i)\sum_{j\neq i}\theta_{ij}p_ip_j \tag{100}$$

$$+ \{\text{terms constant in } p_i\}$$

$$= (1+2s_ic_i)p_i\left\{Q_i\left(1-\frac{p_i}{2p_i^0}\right) + \sum_{j\neq i}\theta_{ij}p_j\right\} \tag{101}$$

$$+ \{\text{terms constant in } p_i\}.$$

Therefore,

$$u_i(p_i', p_{-i}) - u_i(p_i'', p_{-i}) = (1+2s_ic_i)Q_i\left\{p_i'\left(1-\frac{p_i'}{2p_i^0}\right) - p_i''\left(1-\frac{p_i''}{2p_i^0}\right)\right\} \tag{102}$$

$$+ (1+2s_ic_i)\sum_{j\neq i}\theta_{ij}\left(p_i'-p_i''\right)p_j.$$

On the other hand,

$$P(p_i', p_{-i}) - P(p_i'', p_{-i}) = \widehat{w}_iQ_i\left\{p_i'\left(1-\frac{p_i'}{2p_i^0}\right) - p_i''\left(1-\frac{p_i''}{2p_i^0}\right)\right\} \tag{103}$$

$$+ \widehat{w}_i\sum_{j=1}^{i-1}\theta_{ij}\left(p_i'-p_i''\right)p_j + \sum_{j=i+1}^{n}\widehat{w}_j\theta_{ji}p_j(p_i'-p_i'').$$

We will show below that

$$\widehat{w}_j\theta_{ji} = \widehat{w}_i\theta_{ij} \qquad (j = i+1, ..., n). \tag{104}$$

Using this in (103), and recalling (102), we get

$$P(p_i', p_{-i}) - P(p_i'', p_{-i}) = \widehat{w}_iQ_i\left\{p_i'\left(1-\frac{p_i'}{2p_i^0}\right) - p_i''\left(1-\frac{p_i''}{2p_i^0}\right)\right\} \tag{105}$$

$$+ \widehat{w}_i\sum_{j\neq i}\theta_{ij}\left(p_i'-p_i''\right)p_j$$

$$= \frac{\widehat{w}_i}{1+2s_ic_i}\left(u_i(p_i', p_{-i}) - u_i(p_i'', p_{-i})\right), \tag{106}$$

as claimed. So it remains to be checked that (104) holds true, or equivalently, that

$$|\theta_{12}|\cdot ... \cdot|\theta_{j-1j}|\cdot|\theta_{j+1j}|\cdot ... \cdot|\theta_{nn-1}|\cdot\theta_{ji}$$

$$= |\theta_{12}|\cdot ... \cdot|\theta_{i-1i}|\cdot|\theta_{i+1i}|\cdot ... \cdot|\theta_{nn-1}|\cdot\theta_{ij} \qquad (j = i+1, ..., n). \tag{107}$$

But this was shown in the proof of Theorem 1. Since $\widehat{w}_i > 0$ for $i \in N$, we conclude that P is indeed a weighted potential for the differentiated Bertrand game. The proposition follows. □

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