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# Revenue Ranking of Optimally Biased Contests: The Case of Two Players

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# Revenue Ranking of Optimally Biased Contests: The Case of Two Players*

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**Abstract** It is shown that the equilibrium in the asymmetric Tullock contest is unique for parameter values $r \leq 2$. This allows proving a *revenue ranking result* saying that a revenue-maximizing designer capable of biasing the contest always prefers a contest technology with higher accuracy.

**Keywords** Tullock contest · Nash equilibrium · Heterogeneous valuations · Discrimination

**JEL Classification** C72 · D72 · J71

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## 1. Introduction

Contests are used widely in economics and political theory. Specific applications include marketing, rent-seeking, campaigning, military conflict, and sports, for instance.[1] A useful contest technology, conveniently parameterized by a parameter $r \in (0, \infty)$, has been popularized by Tullock (1980). Pure-strategy Nash equilibria have been identified for low values of $r$ (Mills, 1959; Pérez-Castrillo and Verdier, 1992; Nti, 1999, 2004; Cornes and Hartley, 2005), and mixed-strategy equilibria for high values of $r$ (Baye et al., 1994; Alcade and Dahm, 2010; Ewerhart, 2015, 2016). For intermediate values of $r$ and heterogeneous valuations, Wang (2010) has constructed additional equilibria in which only one player randomizes.

The present paper complements and, in a sense, completes the equilibrium analysis of Tullock's model in the important special case of two players and heterogeneous valuations. We first show that, for $r \leq 2$, the equilibrium is unique. This observation is useful because for $r > 2$, the usual equilibrium characteristics, such as expected efforts, participation probabilities, winning probabilities, expected payoffs, and revenue, are known to be independent of the equilibrium. Then, we document the properties of the equilibrium, including rent-dissipation, comparative statics, and robustness. Finally, as an application, we prove a revenue ranking result for optimally biased contests.

The remainder of this paper is structured as follows. Section 2 introduces the notation and reviews existing equilibrium characterizations. Section 3 presents our uniqueness result. Comparative statics are discussed in Section 4. Section 5 deals with robustness. Optimal discrimination is examined in Section 6. An Appendix contains an auxiliary result.

## 2. Set-up and notation

There are two players $i = 1, 2$. Player $i$'s valuation of the prize is denoted by $V_i$, where we assume $V_1 \geq V_2 > 0$. Given efforts $x_1 \geq 0$ for player 1 and $x_2 \geq 2$ for player 2, player $i$'s probability of winning is specified as

$$p_i(x_1, x_2) = \frac{x_i^r}{x_1^r + x_2^r}, \tag{1}$$

where $r \in (0, \infty)$, and the ratio is replaced by $p_i^0 = 0.5$ should the denominator vanish.[2] Player $i$'s payoff is given by $\Pi_i = p_i V_i - x_i$. This defines the **two-player contest** $\mathcal{C} = \mathcal{C}(V_1, V_2, r)$.

A **mixed strategy** $\mu_i$ for player $i$ is probability measure on $[0, V_i]$. Let $\mathcal{M}_i$ denote the set of player $i$'s mixed strategies. Given $\mu = (\mu_1, \mu_2) \in \mathcal{M}_1 \times \mathcal{M}_2$, we write $p_i(\mu_1, \mu_2) = E[p_i(x_1, x_2)|\, \mu]$ and

[1] Cf. Konrad (2009).

[2] The assumption on $p_i^0$ will be relaxed in Section 5.

$\Pi_i(\mu_1, \mu_2) = E[\Pi_i(x_1, x_2)|\,\mu]$, where $E[\cdot|\,\mu]$ denotes the expectation operator. An **equilibrium** is a pair $\mu^* = (\mu_1^*, \mu_2^*) \in \mathcal{M}_1 \times \mathcal{M}_2$ satisfying $\Pi_1(\mu_1^*, \mu_2^*) \geq \Pi_1(\mu_1, \mu_2^*)$ for any $\mu_1 \in \mathcal{M}_1$ and $\Pi_2(\mu_1^*, \mu_2^*) \geq \Pi_2(\mu_1^*, \mu_2)$ for any $\mu_2 \in \mathcal{M}_2$.

For an equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$, we define player $i$'s **expected effort** $\overline{x}_i = E[x_i|\,\mu_i^*]$, **participation probability** $\pi_i = \mu_i^*(\{x_i > 0\})$, **winning probability** $p_i^* = p_i(\mu_1^*, \mu_2^*)$, and **expected payoff** $\Pi_i^* = p_i^* V_i - \overline{x}_i$, as well as the designer's **revenue** $\mathcal{R} = \overline{x}_1 + \overline{x}_2$. An equilibrium $\mu^*$ is an **all-pay auction equilibrium** if it shares these characteristics with the unique equilibrium of the corresponding all-pay auction (Alcade and Dahm, 2010).

Let $\omega = V_2/V_1$. The following three propositions summarize much of the existing equilibrium characterizations.

**Proposition 1. (Mills, 1959; Pérez-Castrillo and Verdier, 1992; Nti, 1999, 2004; Cornes and Hartley, 2005)** *A pure-strategy equilibrium exists if and only if* $r \leq 1 + \omega^r$. *This equilibrium is interior, and unique within the class of pure-strategy equilibria.*[3]

**Proposition 2. (Baye et al., 1994; Alcade and Dahm, 2010; Ewerhart, 2015, 2016)** *For any* $r \geq 2$, *there exists an all-pay auction equilibrium. Moreover, for* $r > 2$, *any equilibrium is an all-pay auction equilibrium, and both players randomize.*

**Proposition 3 (Alcade and Dahm, 2010; Wang, 2010).** *For any* $r \in (1 + \omega^r, 2]$, *there exists an equilibrium in which player 1 chooses a pure strategy, while player 2 randomizes between zero and a positive effort.*

For convenience, the cases captured by Propositions 1 through 3, respectively, will be referred to as the pure, mixed, and semi-mixed cases. See Figure 1 for illustration.[4]

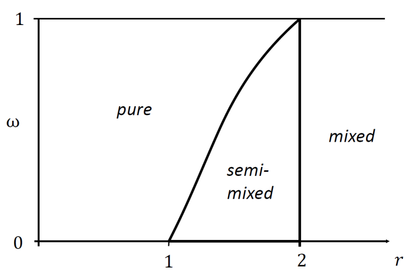


Figure 1: The parameter space.

[3] For homogeneous valuations and $r \leq 2$, the equilibrium is known to be unique even within the class of all equilibria.

[4] Note the overlap between the cases. Indeed, for $r = 2$ and $\omega = 1$, the all-pay auction equilibrium is in pure strategies. Further, for $r = 2$ and $\omega < 1$, the semi-mixed equilibrium is an all-pay auction equilibrium.

## 3. Uniqueness

The following result is key to all what follows.

**Proposition 4.** *For any $r \leq 2$, there is precisely one equilibrium.*

**Proof.** Assume first that $r \leq 1 + \omega^r$. By Proposition 1, there exists an interior pure-strategy equilibrium $(x_1^*, x_2^*)$. Moreover, the only candidate for an alternative best response to $x_1^*$ is the zero bid (Pérez-Castrillo and Verdier, 1992; Cornes and Hartley, 2005). Since equilibria in contests are interchangeable (cf. the Appendix), the support of any alternative equilibrium strategy must be a subset of $\{0, x_2^*\}$. However, player 1's first-order necessary condition for the interior optimum,

$$\frac{\partial p_1(x_1^*, x_2^*)}{\partial x_1} V_1 \pi_2 - 1 = 0, \tag{2}$$

holds for $\pi_2 = 1$, so that it cannot hold for $\pi_2 < 1$. By an analogous argument, necessarily $\pi_1 = 1$ and, hence, the equilibrium is unique in this case. Assume next that $r > 1 + \omega^r$. By Proposition 3, there exists a semi-mixed equilibrium in which player 1 uses a pure strategy $x_1^* > 0$, while player 2 randomizes, choosing some $x_2 = x_2^*$ with probability $\pi_2 \in (0, 1)$, and $x_2 = 0$ otherwise. As above, it follows that player 2's best-response set is $\{0, x_2^*\}$. Any alternative equilibrium strategy could, therefore, only use a different probability $\pi_2$ of randomization across the set $\{0, x_2^*\}$. But this is impossible in view of (2), which must hold also in the semi-mixed case. Moreover, by the construction of the semi-mixed equilibrium (Alcade and Dahm, 2010; Wang, 2010), player 1's best-response set is the same as in the associated pure-strategy equilibrium in the contest $\widehat{\mathcal{C}} = \mathcal{C}(\widehat{V}_1, V_2, r)$, with $\widehat{V}_1 = V_2/(1-r)^{1/r}$. Hence, $x_1^*$ is the unique best response, and uniqueness of the equilibrium follows as above. □

Proposition 4 implies, in particular, that for $r = 2$, there does not exist any equilibrium other than the all-pay auction equilibrium identified by Alcade and Dahm (2010, Ex. 3.3).[5]

Define **rent dissipation** as the fraction $\phi_i = \overline{x}_i / V_i$ of the valuation spent by player $i$. In the pure and mixed cases, $\phi_i$ is known to be identical for the two players, with $\phi \equiv \phi_1 = \phi_2$ being strictly increasing in $\omega$. As noted by Wang (2010), this extends to the semi-mixed case, where

$$\phi = \alpha(r) \frac{\omega}{2}, \tag{3}$$

[5] Unfortunately, however, the argument does not deliver uniqueness for $r > 2$ because the best-response set is countably infinite in that case.

with

$$\alpha(r) = \frac{2}{r}(r-1)^{\frac{r-1}{r}}. \tag{4}$$

The present analysis shows that $\phi$ is globally strictly increasing in $\omega$ for any $r \in (0, \infty)$, regardless of the equilibrium.

## 4. Comparative statics

Table I provides an overview of the comparative statics of the equilibrium.[6] As can be seen, the comparative statics of the semi-mixed equilibrium with respect to $V_1$ and $V_2$ is identical to that of the all-pay auction. The comparative statics of the semi-mixed equilibrium with respect to $r$ is as follows. As the contest becomes more decisive, expected efforts, player 2's participation probability, and revenue are all strictly declining towards the respective all-pay auction levels. In contrast, player 1's winning probability and expected payoff are both strictly increasing towards the respective all-pay auction levels.

| **pure** | | | | **semi-mixed** | | | | **mixed** | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $r \leq 1 + \frac{V_2^r}{V_1^r}$ | | | | $r \in (1 + \frac{V_2^r}{V_1^r}, 2]$ | | | | $r > 2$ | | |
| | $dV_1$ | $dV_2$ | $dr$ | | $dV_1$ | $dV_2$ | $dr$ | | $dV_1$ | $dV_2$ |
| $x_1^* = \frac{rV_1^{r+1}V_2^r}{(V_1^r+V_2^r)^2}$ | $+$ | $+$ | $\pm$ | $x_1^* = \alpha(r)\frac{V_2}{2}$ | 0 | $+$ | $-$ | $\overline{x}_1 = \frac{V_2}{2}$ | 0 | $+$ |
| $x_2^* = \frac{rV_1^rV_2^{r+1}}{(V_1^r+V_2^r)^2}$ | $-$ | $+$ | $\pm$ | $\overline{x}_2 = \alpha(r)\frac{V_2^2}{2V_1}$ | $-$ | $+$ | $-$ | $\overline{x}_2 = \frac{V_2^2}{2V_1}$ | $-$ | $+$ |
| $\pi_2 = 1$ | 0 | 0 | 0 | $\pi_2 = \frac{V_2}{V_1(r-1)^{1/r}}$ | $-$ | $+$ | $-$ | $\pi_2 = \frac{V_2}{V_1}$ | $-$ | $+$ |
| $p_1^* = \frac{V_1^r}{V_1^r+V_2^r}$ | $+$ | $-$ | $+$ | $p_1^* = 1 - \alpha(r)\frac{V_2}{2V_1}$ | $+$ | $-$ | $+$ | $p_1^* = 1 - \frac{V_2}{2V_1}$ | $+$ | $-$ |
| $\Pi_1^* = \frac{V_1^{r+1}(V_1^r+V_2^r-rV_2^r)}{(V_1^r+V_2^r)^2}$ | $+$ | $-$ | $\pm$ | $\Pi_1^* = V_1 - \alpha(r)V_2$ | $+$ | $-$ | $+$ | $\Pi_1^* = V_1 - V_2$ | $+$ | $-$ |
| $\Pi_2^* = \frac{V_2^{r+1}(V_1^r+V_2^r-rV_1^r)}{(V_1^r+V_2^r)^2}$ | $-$ | $+$ | $-$ | $\Pi_2^* = 0$ | 0 | 0 | 0 | $\Pi_2^* = 0$ | 0 | 0 |
| $\mathcal{R} = \frac{rV_1^rV_2^r(V_1+V_2)}{(V_1^r+V_2^r)^2}$ | $\pm$ | $+$ | $\pm$ | $\mathcal{R} = \alpha(r)(\frac{V_2}{2} + \frac{V_2^2}{2V_1})$ | $-$ | $+$ | $-$ | $\mathcal{R} = \frac{V_2}{2} + \frac{V_2^2}{2V_1}$ | $-$ | $+$ |

Table I: Comparative statics.

One can check that all the equilibrium characteristics listed in the table depend continuously on parameters. In other words, there are no jumps in the possible transitions between pure, semi-mixed, and mixed equilibria. This allows deriving global comparative statics results. For example, Yildirim (2015) has made the intuitive observation that, if the contest technology exhibits decreasing returns, the weaker player never prefers a more decisive contest. But, as $d\Pi_2^*/dr \leq 0$ holds globally, the same conclusion holds for technologies with constant or increasing returns.

[6] The table summarizes and extends the results of Nti (1999, 2004), Wang (2010), and Yildirim (2015).

## 5. Robustness

So far, we assumed that $p_1^0 = p_2^0 = 0.5$. However, as our next result shows, this assumption is not crucial.

**Proposition 5.** *The equilibrium set remains unchanged when* $p_1^0, p_2^0 \in [0,1]$ *and* $p_1^0 + p_2^0 \leq 1$.

**Proof.** Let $\mu^* = (\mu_1^*, \mu_2^*)$ be an equilibrium under the modified rules that is not an equilibrium in $\mathcal{C}$. Since mutual inactivity cannot occur with positive probability in $\mu^*$, some player $i$ finds a deviation to zero profitable in $\mathcal{C}$, but not profitable under the modified rules. Moreover, $\mu_j^*$, with $j \neq i$, must have an atom at zero, and $p_i^0 < 0.5$. But then, player $i$ has a profitable deviation to some small $x_i > 0$ both in $\mathcal{C}$ and under the modified rules. Contradiction! Conversely, suppose that $\mu^* = (\mu_1^*, \mu_2^*)$ is an equilibrium in $\mathcal{C}$ that is not an equilibrium under the modified rules. Then some player $i$ finds a deviation to zero profitable under the modified rules, yet not profitable in $\mathcal{C}$. Moreover, player $j$'s mixed strategy $\mu_j^*$, with $j \neq i$, necessarily has an atom at zero. Given Propositions 1 and 4, this is feasible only if $i = 1$ and $r > 1 + \omega^r$. In the semi-mixed case, however, bidding zero yields a payoff for player 1 of

$$\Pi_1 = p_1^0 V_1 (1 - \pi_2) \leq V_1 (1 - \pi_2) = V_1 - \frac{V_2}{(r-1)^{1/r}}, \tag{5}$$

which is weakly less than

$$\Pi_1^* = \left\{ \pi_2 \frac{(x_1^*)^r}{(x_1^*)^r + (x_2^*)^r} + 1 - \pi_2 \right\} V_1 - x_1^* = V_1 - \frac{2(r-1)^{\frac{r-1}{r}}}{r} V_2, \tag{6}$$

because $2(r-1)/r \leq 1$. Similarly, in the mixed case, $\Pi_1^* = V_1 - V_2$, whereas a deviation to zero yields only $\Pi_1 = p_1^0 V_1 (1 - \pi_2) \leq V_1 - V_2$. Contradiction! □

## 6. Optimally biased contests

Suppose now that a designer may inflate or deflate player 2's effort by a factor $\lambda > 0$. I.e., players' probabilities of winning are given in the interior by

$$p_1^\lambda (x_1, x_2) = \frac{x_1^r}{x_1^r + (\lambda x_2)^r} \tag{7}$$

and $p_2^\lambda = 1 - p_1^\lambda$. Let $\phi(\lambda)$ denote the rent-dissipation in the contest $\mathcal{C}^\lambda = \mathcal{C}(V_1^\lambda, V_2^\lambda, r)$, where $V_1^\lambda = \max\{V_1, \lambda V_2\}$ and $V_2^\lambda = \min\{V_1, \lambda V_2\}$. Since players act *as if* in $\mathcal{C}^\lambda$, the **revenue from the**

**biased contest** is given by

$$\mathcal{R}(\lambda) = (V_1 + V_2)\phi(\lambda). \tag{8}$$

Franke et al. (2014) obtained a dominance result. Epstein et al. (2013) compared pure-strategy equilibria directly with all-pay auction equilibria. The following result ranks a continuum of contest technologies, explicitly taking into account the possibility of semi-mixed equilibria.

**Proposition 6.** *For any $r \in (0, \infty)$, the revenue-maximizing bias is $\lambda^* = 1/\omega$, with*

$$\mathcal{R}(\lambda^*) = \min\{\frac{r}{2}, 1\} \cdot \frac{V_1 + V_2}{2} \tag{9}$$

*Thus, the revenue from the optimally biased contest is strictly increasing for $r \leq 2$, and constant for $r \geq 2$.*

**Proof.** Suppose first that $r \leq 2$. In a pure-strategy equilibrium, maximizing

$$\mathcal{R}(\lambda) = \frac{rV_1^r(\lambda V_2)^r(V_1 + V_2)}{(V_1^r + \lambda^r V_2^r)^2} \tag{10}$$

yields the solution $\lambda^* = 1/\omega$, with $\mathcal{R}(\lambda^*) = (r/4) \cdot (V_1 + V_2)$. For a semi-mixed equilibrium,

$$\mathcal{R}(\lambda) = \begin{cases} \frac{\lambda\omega}{2}\alpha(r)(V_1 + V_2) & \text{if } \lambda\omega < (r-1)^{1/r} \\ \frac{1}{2\lambda\omega}\alpha(r)(V_1 + V_2) & \text{if } \lambda\omega > (r-1)^{-1/r}. \end{cases} \tag{11}$$

In the first case, $\mathcal{R}(\lambda)$ is strictly increasing in $\lambda$. In the second case, $\mathcal{R}(\lambda)$ is strictly declining in $\lambda$. Hence, it is strictly suboptimal to implement a semi-mixed equilibrium. For $r > 2$, the claim has been proved by the author in prior work (2016). □

## Appendix A. An auxiliary result

Two equilibria $(\mu_1^*, \mu_2^*)$ and $(\mu_1^{**}, \mu_2^{**})$ are called **interchangeable** if $(\mu_1^*, \mu_2^{**})$ and $(\mu_1^{**}, \mu_2^*)$ are equilibria as well.

**Lemma A.1.** *Equilibria in two-player contests are interchangeable.*

**Proof.** The proof is a straightforward adaptation on an argument detailed in Klumpp and Polborn (2006, p. 1104), and therefore omitted. □

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