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Unique Equilibrium in Rent-Seeking Contests with a Continuum of Types

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Abstract It is shown that rent-seeking contests with continuous and independent type distributions possess a unique pure-strategy Nash equilibrium.

Keywords Rent-seeking · Private information · Pure-strategy Nash equilibrium · Existence · Uniqueness

JEL Classification C7, D7, D8

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1 Introduction

While rent-seeking contests with continuous and independent type distributions are quite interesting, basic issues such as existence and uniqueness of a pure-strategy Nash equilibrium (PSNE) have been addressed only partially. Indeed, previous work on the issue of existence focused either on symmetric contests (Fey, 2008; Ryvkin, 2010) or on the case of a continuous technology (Wasser, 2013a, 2013b). Moreover, little general was known about the uniqueness of the equilibrium.

Below, it is shown that in any rent-seeking contest with independent and continuous types, there exists a unique PSNE.[1] The result holds even when the contest is ex-ante asymmetric,[2] so that the equilibrium may entail inactive types.[3] Moreover, no restriction is imposed on the shape of the type distributions. Generally, existence ensures consistency of a model, whereas uniqueness strengthens numerical analyses, theoretical results, and experimental findings.

The rest of the paper is structured as follows. Section 2 describes the set-up. Existence is dealt with in Section 3. Section 4 discusses uniqueness. A numerical illustration can be found in Section 5. Section 6 concludes. An Appendix contains technical lemmas.

[1] Uniqueness means here that for any given player, any two PSNE strategies differ at most on a null set. This corresponds to the strongest form of uniqueness for PSNE.

[2] Asymmetry may be reflected, e.g., in heterogeneous distributions of marginal costs or in heterogeneous economies of scale.

[3] Wärneryd (2003) explicitly allows for inactive types in a common-value setting.

2 Set-up

There are $N \geq 2$ players. Each player $i = 1, ..., N$ observes a signal (or type) c_i, drawn from an interval $D_i = [\underline{c}_i, \bar{c}_i]$, where $0 < \underline{c}_i < \bar{c}_i$. Signals are independent across players. Moreover, player i does not observe the signal c_j of any other player $j \neq i$. The distribution function of player i's signal is denoted by $F_i = F_i(c_i)$. Each player i chooses a level of activity $y_i \geq 0$ at cost $g_i(y_i)$. It is assumed that $g_i(0) = 0$, and that g_i is twice continuously differentiable on $\mathbb{R}_+$, with $g_i' > 0$ on $\mathbb{R}_{++}$, and $g_i'' \geq 0$. Player i's payoff is $\Pi_i(y_i, y_{-i}, c_i) = p_i(y_i, y_{-i}) - c_i g_i(y_i)$, where $p_i(y_i, y_{-i}) = y_i/(y_i + \sum_{j \neq i} y_j)$ if $y_i + \sum_{j \neq i} y_j > 0$, and $p_i(y_i, y_{-i}) = 1/N$ otherwise.[4]

A strategy for player i is a (measurable) mapping $\sigma_i : D_i \to \mathbb{R}_+$. Denote by S_i the set of strategies for player i. For a profile $\sigma_{-i} = \{\sigma_j\}_{j \neq i} \in S_{-i} = \prod_{j \neq i} S_j$, and a type $c_i \in D_i$, player i's interim expected payoff is given by $\overline{\Pi}_i(y_i, \sigma_{-i}, c_i) = \int_{D_{-i}} \Pi_i(y_i, \sigma_{-i}(c_{-i}), c_i) dF_{-i}(c_{-i})$, where $D_{-i} = \prod_{j \neq i} D_j$, $\sigma_{-i}(c_{-i}) = \{\sigma_j(c_j)\}_{j \neq i}$, and $dF_{-i}(c_{-i}) = \prod_{j \neq i} dF_j(c_j)$. A *Bayesian Nash equilibrium (BNE)* is a profile $\sigma^* = \{\sigma_i^*\}_{i=1}^N \in S = \prod_{i=1}^N S_i$ such that $\overline{\Pi}_i(\sigma_i^*(c_i), \sigma_{-i}^*, c_i) \geq \overline{\Pi}_i(y_i, \sigma_{-i}^*, c_i)$ for any $i = 1, ..., N$, any $c_i \in D_i$, and any $y_i \geq 0$. A *pure-strategy Nash equilibrium (PSNE)* is a profile $\sigma^* \in S$ such that for any $i = 1, ..., N$, and for almost any $c_i \in D_i$, the inequality $\overline{\Pi}_i(\sigma_i^*(c_i), \sigma_{-i}^*, c_i) \geq \overline{\Pi}_i(y_i, \sigma_{-i}^*, c_i)$ holds for any $y_i \geq 0$.[5]

[4] As usual, a simple change of variables allows to capture other types of contest success functions and other forms of uncertainty, e.g., about valuations. Cf. Ryvkin (2010).

[5] As shown in the Appendix, this amounts to the standard definition.

3 Existence

This section builds on prior work by Fey (2008), Ryvkin (2010), and Wasser (2013a). Existence is shown first for the ε-*constrained contest*, for $\varepsilon > 0$, in which each player $i = 1, ..., N$ may use only strategies with values in $[\varepsilon, \infty)$.

Lemma 3.1 *There is a level of activity $E > 0$ such that, for any sufficiently small $\varepsilon > 0$, there exists a BNE σ^ε in the ε-constrained contest such that each player i's strategy σ_i^ε is continuous, monotone, and bounded by E.*

Proof. Since costs are strictly increasing and convex, there is an $E > 0$ such that any $y_i > E$ is suboptimal. Moreover, $\overline{\Pi}_i$ exhibits decreasing differences in y_i and c_i. Hence, existence of a monotone PSNE $\widetilde{\sigma}^\varepsilon$ in the ε-constrained contest follows from Athey (2001, Cor. 2.1). Note now that type c_i's ε-constrained problem, $\max_{y_i \geq \varepsilon} \overline{\Pi}_i(y_i, \widetilde{\sigma}_{-i}^\varepsilon, c_i)$, has a unique solution $y_i = \sigma_i^\varepsilon(c_i)$. Indeed, if $\widetilde{\sigma}_{-i}^\varepsilon(c_{-i}) \neq 0$ with positive probability, then $\overline{\Pi}_i(\cdot, \widetilde{\sigma}_{-i}^\varepsilon, c_i)$ is strictly concave on $[\varepsilon, E]$, while otherwise, the unique solution is $y_i = \varepsilon$. Hence, $\sigma_i^\varepsilon(c_i) = \widetilde{\sigma}_i^\varepsilon(c_i)$ with probability one, for any $i = 1, ..., N$. This implies that $\sigma_i^\varepsilon(c_i)$ is also type c_i's best response to σ_{-i}^ε, for any $i = 1, ..., N$, and any $c_i \in D_i$. Thus, $\sigma^\varepsilon = (\sigma_1^\varepsilon, ..., \sigma_N^\varepsilon)$ is a BNE in the ε-constrained contest. Clearly, each σ_i^ε is monotone. Finally, continuity of σ_i^ε follows from Berge's Theorem, as $\overline{\Pi}_i(\cdot, \sigma_{-i}^\varepsilon, \cdot)$ is continuous on the compact set $[\varepsilon, E] \times D_i$. $\square$

Consider now a sequence $\{\varepsilon_m\}_{m=1}^\infty$ such that $\varepsilon_m \searrow 0$, and select a BNE σ^m in the ε_m-constrained contest for each $m \in \mathbb{N}$, with the properties specified in the previous lemma.

Lemma 3.2 *The sequence $\{\sigma^m\}_{m=1}^\infty$ has a uniformly converging subse-*

quence.

Proof. In view of Lemma 3.1 and the Theorem of Arzelà-Ascoli, it suffices to find a $\lambda > 0$ such that σ_i^m has everywhere a slope exceeding $-\lambda$ for any $m \in \mathbb{N}$ and any i. In terms of the transformed choice variable $y_i^\lambda = y_i + \lambda c_i$, a type c_i's expected payoff in σ^m may be written as

$$\overline{\Pi}_i^\lambda(y_i^\lambda, \sigma_{-i}^m, c_i) = \int_{D_{-i}} \frac{(y_i^\lambda - \lambda c_i) dF_{-i}(c_{-i})}{y_i^\lambda - \lambda c_i + \sum_{j \neq i} \sigma_j^m(c_j)} - c_i g_i(y_i^\lambda - \lambda c_i), \tag{1}$$

provided that $y_i^\lambda - \lambda c_i = y_i > 0$. Hence, for λ sufficiently large, the cross-partial

$$\begin{aligned} \frac{\partial^2 \overline{\Pi}_i^\lambda}{\partial y_i^\lambda \partial c_i} &= \int_{D_{-i}} \frac{2\lambda \sum_{j \neq i} \sigma_j^m(c_j) dF_{-i}(c_{-i})}{\left(y_i + \sum_{j \neq i} \sigma_j^m(c_j)\right)^3} - g_i'(y_i) + \underbrace{c_i \lambda g_i''(y_i)}_{\geq 0} && (2) \\ &\geq \frac{2\lambda}{NE} \int_{D_{-i}} \frac{\sum_{j \neq i} \sigma_j^m(c_j) dF_{-i}(c_{-i})}{\left(y_i + \sum_{j \neq i} \sigma_j^m(c_j)\right)^2} - g_i'(y_i) && (3) \\ &\geq \left(\frac{2\lambda \underline{c}_i}{NE} - 1\right) g_i'(y_i) && (4) \end{aligned}$$

is seen to be positive in the range of c_i where $y_i = \sigma_i^m(c_i) > 0$. Thus, for λ large, y_i^λ is weakly increasing in c_i, which proves the claim. $\square$

By Lemma 3.2, one may assume that $\{\sigma^m\}_{m=1}^\infty$ converges uniformly to some $\sigma^* \in S$. Next, it is shown that in σ^*, at least one player is active with probability one.

Lemma 3.3 *There is some player i such that $\sigma_i^*(c_i) > 0$ with probability one.*

Proof. Suppose that for each i, there is a set $\mathcal{D}_i \subseteq D_i$ of positive measure such that $\sigma_i^*(c_i) = 0$ for all $c_i \in \mathcal{D}_i$. Then, by uniform convergence, there exists, for any $\varepsilon > 0$, an $m_0 = m_0(\varepsilon)$ such that $\sigma_i^m(c_i) < \varepsilon$ for any i, any $c_i \in \mathcal{D}_i$, and any $m \geq m_0$. But, from the Kuhn-Tucker condition for type c_i in the ε_m-constrained contest,

$$0 \geq \int_{\mathcal{D}_{-i}} \frac{\sum_{j \neq i} \sigma_j^m(c_j) dF_{-i}(c_{-i})}{\left(\sigma_i^m(c_i) + \sum_{j \neq i} \sigma_j^m(c_j)\right)^2} - c_i g_i'(E), \tag{5}$$

where $\mathcal{D}_{-i} = \prod_{j \neq i} \mathcal{D}_j$. Integrating over $\mathcal{D}_i$, and subsequently summing over $i = 1, ..., N$, one obtains

$$0 \geq \int_{\mathcal{D}} \frac{(N-1)dF(c)}{\sum_{i=1}^N \sigma_i^m(c_i)} - \sum_{i=1}^N g_i'(E) \int_{\mathcal{D}_i} c_i dF_i(c_i), \tag{6}$$

where $\mathcal{D} = \prod_{i=1}^N \mathcal{D}_i$ and $dF(c) = \prod_{i=1}^N dF_i(c_i)$. For ε small, however, this is impossible. $\square$

The following is the first main result of this paper.

Theorem 3.4 *In the unconstrained contest, σ^* is a PSNE in continuous and monotone strategies.*

Proof. Fix a player $i \in \{1, ..., N\}$. For any $m \in \mathbb{N}$, since σ^m is a BNE in the ε_m-constrained contest, $\overline{\Pi}_i(\sigma_i^m(c_i), \sigma_{-i}^m, c_i) \geq \overline{\Pi}_i(y_i, \sigma_{-i}^m, c_i)$ for any $c_i \in D_i$ and any $y_i \geq \varepsilon_m$. Therefore, if the event $\sigma_{-i}^*(c_{-i}) = 0$ is null, letting $m \to \infty$ implies $\overline{\Pi}_i(\sigma_i^*(c_i), \sigma_{-i}^*, c_i) \geq \overline{\Pi}_i(y_i, \sigma_{-i}^*, c_i)$ for any $c_i \in D_i$ and any $y_i > 0$. Suppose next that $\sigma_{-i}^*(c_{-i}) = 0$ with positive probability. Then, by Lemma 3.3, $\sigma_i^*(c_i) > 0$ with probability one. Let $c_i \in D_i$ with

$\sigma_i^*(c_i) > 0$. If $y_i > 0$, then the argument proceeds as above. To complete the proof, note that $\overline{\Pi}_i(\cdot, \sigma_{-i}^*, c_i)$ is l.s.c., so that $y_i = 0$ cannot be the only profitable deviation for c_i. $\square$

4 Uniqueness

Consider two PSNE σ^* and σ^{**} such that, for some player i, the event $\sigma_i^*(c_i) \neq \sigma_i^{**}(c_i)$ has positive probability. Then, as noted below, σ^* and σ^{**} must differ in an essential way for at least two players.

Lemma 4.1 *There are players $i \neq j$ such that each of the independent events $\sigma_i^*(c_i) \neq \sigma_i^{**}(c_i)$ and $\sigma_j^*(c_j) \neq \sigma_j^{**}(c_j)$ has positive probability.*

Proof. Suppose there is some i such that $\sigma_{-i}^*(c_{-i}) = \sigma_{-i}^{**}(c_{-i})$ with probability one. Then, $\overline{\Pi}_i(\cdot, \sigma_{-i}^*, c_i) = \overline{\Pi}_i(\cdot, \sigma_{-i}^{**}, c_i)$ for any $c_i \in D_i$. Thus, $\sigma_i^*(c_i) = \sigma_i^{**}(c_i)$ with probability one, which is a contradiction. $\square$

The following is the second main result of this paper.

Theorem 4.2 *The PSNE in the unconstrained contest is unique.*

Proof. Following Rosen (1965), write $\sigma^{*,s} = (1-s)\sigma^* + s\sigma^{**}$ for $0 \leq s \leq 1$, and consider

$$\Phi_s = \sum_{i=1}^{N} \int_{D_i} \overline{\pi}_i(\sigma^{*,s}, c_i) \left(\sigma_i^{**}(c_i) - \sigma_i^*(c_i)\right) dF_i(c_i) \tag{7}$$

for $s = 0, 1$, where $\overline{\pi}_i(\sigma, c_i) = \partial\overline{\Pi}_i(\sigma_i(c_i), \sigma_{-i}, c_i)/\partial y_i$ denotes type c_i's marginal expected payoff at a profile $\sigma \in S$.[6] From the Kuhn-Tucker con-

[6] It is shown in the Appendix that Φ_0 and Φ_1 are well-defined.

ditions, $\overline{\pi}_i(\sigma^*, c_i) \leq 0$ for almost any $c_i \in D_i$; moreover, $\sigma_i^*(c_i) = 0$ if $\overline{\pi}_i(\sigma^*, c_i) < 0$. It follows that $\Phi_0 \leq 0$, and similarly, $\Phi_1 \geq 0$. To provoke a contradiction, it will be shown now that $\Phi_1 - \Phi_0 < 0$. Denote by $\pi_i(\sigma, c_i, c_{-i}) = \partial \Pi_i(\sigma_i(c_i), \sigma_{-i}(c_{-i}), c_i)/\partial y_i$ type c_i's marginal ex-post payoff at $\sigma \in S$, when facing $c_{-i} \in D_{-i}$. Then, by Lemma A.2 in the Appendix,

$$\begin{aligned} \Phi_1 - \Phi_0 &= \int_D \sum_{i=1}^N (\pi_i(\sigma^{**}, c_i, c_{-i}) - \pi_i(\sigma^*, c_i, c_{-i})) z_i(c_i) dF(c) \qquad (8) \\ &= \int_D \sum_{i=1}^N \left\{ \int_0^1 \frac{\partial \pi_i(\sigma^{*,s}, c_i, c_{-i})}{\partial s} z_i(c_i) ds \right\} dF(c), \qquad (9) \end{aligned}$$

where $z_i(c_i) = \sigma_i^{**}(c_i) - \sigma_i^*(c_i)$. An application of the chain rule delivers

$$\frac{\partial \pi_i(\sigma^{*,s}, c_i, c_{-i})}{\partial s} = \sum_{j=1}^N \frac{\partial^2 p_i(\sigma_i^{*,s}(c_i), \sigma_{-i}^{*,s}(c_{-i}))}{\partial y_i \partial y_j} z_j(c_j) - c_i \underbrace{g_i''(\sigma_i^{*,s}(c_i))}_{\geq 0} z_i(c_i), \qquad (10)$$

for any i, any $c_i \in D_i$, and any $c_{-i} \in D_{-i}$. It follows that

$$\Phi_1 - \Phi_0 \leq \int_D \left(\int_0^1 \left(\sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 p_i(\sigma_i^{*,s}(c_i), \sigma_{-i}^{*,s}(c_{-i}))}{\partial y_i \partial y_j} z_i(c_i) z_j(c_j) \right) ds \right) dF(c). \qquad (11)$$

One can verify, however, that

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2 p_i(y_i,y_{-i})}{\partial y_i \partial y_j} z_i z_j && (12)\\
= \; & -\sum_{i=1}^{N}\frac{2Y_{-i}}{Y^3} z_i^2 + \sum_{i=1}^{N}\sum_{j\neq i}\frac{Y-2Y_{-i}}{Y^3} z_i z_j && (13)\\
= \; & -\frac{2}{Y^3}\sum_{i=1}^{N} Y_{-i} z_i^2 - \frac{2}{Y^3}\sum_{i=1}^{N}\sum_{j>i}\sum_{k\neq i,j} y_k z_i z_j && (14)\\
= \; & -\frac{1}{Y^3}\sum_{i=1}^{N} Y_{-i} z_i^2 - \frac{1}{Y^3}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\neq i,j} y_k z_i z_j && (15)\\
= \; & -\frac{1}{Y^3}\sum_{i=1}^{N}(z_i^2 Y_{-i} + y_i Z_{-i}^2) \leq 0 && (16)
\end{aligned}$$

for any $(y_1, ..., y_N) \in \mathbb{R}_+^N \backslash \{0\}$ and any $(z_1, ..., z_N) \in \mathbb{R}^N$, where $Y = \sum_{i=1}^{N} y_i$, $Y_{-i} = \sum_{j\neq i} y_j$, and $Z_{-i} = \sum_{j\neq i} z_j$. Moreover, $z_i^2 Y_{-i} = z_i(c_i)^2 \sum_{j\neq i} \sigma_j^{*,s}(c_j)$ is positive for any $s \in (0,1)$ if $\sigma_i^*(c_i) \neq \sigma_i^{**}(c_i)$ and $\sigma_j^*(c_j) \neq \sigma_j^{**}(c_j)$ for some $j \neq i$. Thus, by Lemma 4.1, $\Phi_1 - \Phi_0 < 0$. □

5 Numerical illustration

Figure 1 shows PSNE strategies in a two-player lottery contest, where types are distributed uniformly on $D_1 = [0.01, 1.01]$ and $D_2 = [0.51, 5.51]$, respectively. Note that player 2 remains inactive for $c_2 > c_2^* \approx 4.21$.

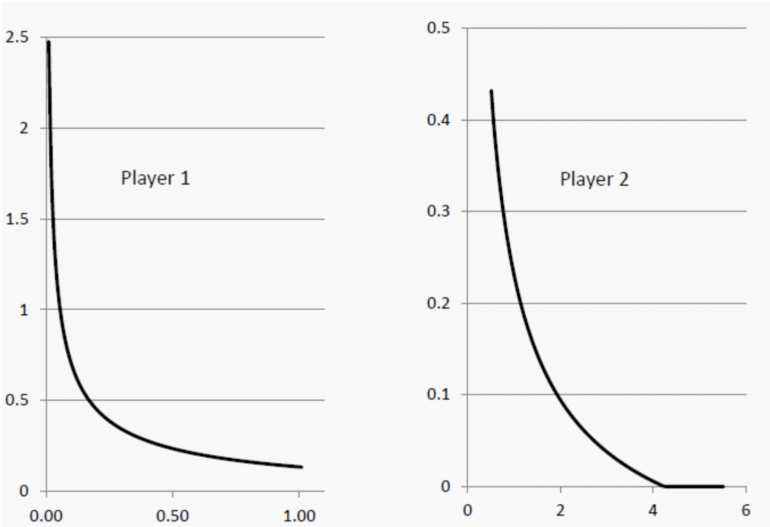

Figure 1: An equilibrium involving inactive types

6 Concluding remark

While this paper has focused on the existence and uniqueness of a PSNE in asymmetric rent-seeking contests, it follows from the proofs that also any of the BNE studied by Fey (2008) and Ryvkin (2010) is unique.

7 Appendix: Technical lemmas

Lemma A.1 *A profile $\sigma^* \in S$ is a PSNE in the unconstrained contest if and only if $\int_D \Pi_i(\sigma_i^*(c_i), \sigma_{-i}^*(c_{-i}), c_i) dF(c) \geq \int_D \Pi_i(\widehat{\sigma}_i(c_i), \sigma_{-i}^*(c_{-i}), c_i) dF(c)$ for any $i = 1, ..., N$, and any $\widehat{\sigma}_i \in S_i$.*

Proof. Let σ^* be a PSNE, and consider a deviation $\widehat{\sigma}_i \in S_i$ for some player i. Then, $\overline{\Pi}_i(\sigma_i^*(c_i), \sigma_{-i}^*, y_i) \geq \overline{\Pi}_i(\widehat{\sigma}_i(c_i), \sigma_{-i}^*, c_i)$ for almost any $c_i \in D_i$. Integrating over D_i, the assertion follows via Fubini's theorem. Conversely, suppose that σ^* is not a PSNE. Then, there is a player i and a set $\mathcal{D}_i \subseteq D_i$ of positive measure such that $\sigma_i^*(c_i)$ is not a best response to σ_{-i}^* for c_i, for any $c_i \in \mathcal{D}_i$. Define $\widehat{\sigma}_i(c_i)$ as c_i's best response to σ_{-i}^* if it exists; otherwise as $\sigma_i^*(c_i)/2$ if $\sigma_i^*(c_i) > 0$, and as $\mathrm{pr}\{\sigma_{-i}^*(c_{-i}) = 0\}/(2\overline{c}_i g_i'(E))$ if $\sigma_i^*(c_i) = 0$. Then $\widehat{\sigma}_i$ is a profitable deviation. $\square$

Lemma A.2 *Let $\sigma^* \in S$ be a PSNE in the unconstrained contest. Then, for almost any $c_i \in D_i$, the function $\pi_i(\sigma^*, c_i, \cdot)$ is integrable, with $\overline{\pi}_i(\sigma^*, c_i) = \int_{D_{-i}} \pi_i(\sigma^*, c_i, c_{-i}) dF_{-i}(c_{-i})$. Moreover, $\overline{\pi}_i(\sigma^*, \cdot)$ is integrable.*

Proof. The first claim is obvious if $\sigma_i^*(c_i) > 0$ for almost any $c_i \in D_i$. Suppose that $\sigma_i^*(c_i) = 0$ with positive probability. Then, by Lemma 3.3, the event $\sigma_{-i}^*(c_{-i}) = 0$ is null. Take some $c_{-i} \in D_{-i}$ with $\sigma_{-i}^*(c_{-i}) \neq 0$. Then, for any $c_i \in D_i$, by concavity, the difference quotient $\Pi_i(y_i, \sigma_{-i}^*(c_{-i}), c_i)/y_i$ is monotone increasing as $y_i \searrow 0$, with limit $\pi_i(\sigma^*, c_i, c_{-i})$. Since also $\Pi_i(y_i, \sigma_{-i}^*(c_{-i}), c_i)/y_i \geq -\overline{c}_i g_i'(E)$, the first claim follows from Levi's theorem. The second claim follows from Lebesgue's theorem, because $\overline{\pi}_i(\sigma^*, \cdot) \leq 0$ from the Kuhn-Tucker conditions, and because $\overline{\pi}_i(\sigma^*, \cdot) \geq -\overline{c}_i g_i'(E)$, as above. $\square$

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