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Money Cycles

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Abstract

Classical models of money are typically based on a competitive market without capital or credit. They then impose exogenous timing structures, market participation constraints, or cash-in-advance constraints to make money essential. We present a simple model without credit where money arises from a fixed cost of production. This leads to a rich equilibrium structure. Agents avoid the fixed cost by taking vacations and the trade between workers and vacationers is supported by money. We show that agents acquire and spend money in cycles of finite length. Throughout such a "money cycle," agents decrease their consumption which we interpret as the hot potato effect of inflation. We give an example where money holdings do not decrease monotonically throughout the money cycle. Optimal monetary policy is given by the Friedman rule, which supports efficient equilibria. Thus, monetary policy provides an alternative to lotteries for smoothing out non-convexities.

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1 Introduction

Classical models of money are typically based on a competitive market without capital or credit. They then impose exogenous timing structures, market participation constraints, or cash-in-advance constraints to give agents a reason to trade and hence make money essential. In the overlapping generations framework of Samuelson (1958), only the young generation may produce and therefore money supports trade between the old and young generations. The model of spatially separated markets of Townsend (1980) uses a particular timing structure where each agent alternates between being a buyer and a seller depending on the stock of endowment. Bewley (1983) uses stochastic endowment shocks and money supports trade between those agents with good and bad shocks. Market participation constraints are used by Wicksell (1906) to guarantee that agents use money to trade in circles. Kiyotaki and Wright (1989) elaborate on Wicksell's approach by replacing the market participation constraints by pairwise random matching. Money then resolves the arising double coincidence of wants problem.

We present a novel reason for fiat money to be essential: a fixed cost of engaging in production in each time period. For example, workers must pay a fixed transport cost before beginning productive activities. Restaurants must prepare fresh ingredients every day they wish to open. Airline pilots must pass security checks before flying. When fixed costs are high, agents do not work every day, and money supports trade between vacationers and workers. This friction gives rise to a rich equilibrium structure. To finance consumption during vacations, agents save by selling surplus production and accumulating money. In order to study the effects of monetary policy, we allow the money stock to change at a constant rate by lump sum transfers. We show that if the money stock expands at a rate above the Friedman rule, then agents' money holdings, consumption and production decisions proceed through repeating *money cycles* of finite length. An agent begins his money cycle with no money, works during the first period and takes a vacation during the final period. Note that money cycles are not attributable to any kind of exogenous shocks or changes in the policy. There is no uncertainty in the model and the monetary policy is stationary. Figure 1 shows an example of a simple money cycle. Money cycles possess several striking features, which we discuss in turn.

Decreasing consumption At the Friedman rule, equilibria are efficient and agents have constant consumption. However, away from the Friedman rule, agents decrease their consumption over the length of the money cycle. This distortion arises because the inflation tax of holding money adds a wedge between the marginal cost and marginal return of saving. We interpret it as the hot potato effect of inflation, because inflation causes agents to trade prematurely. Thus, money is not super-neutral in the money cycles model.

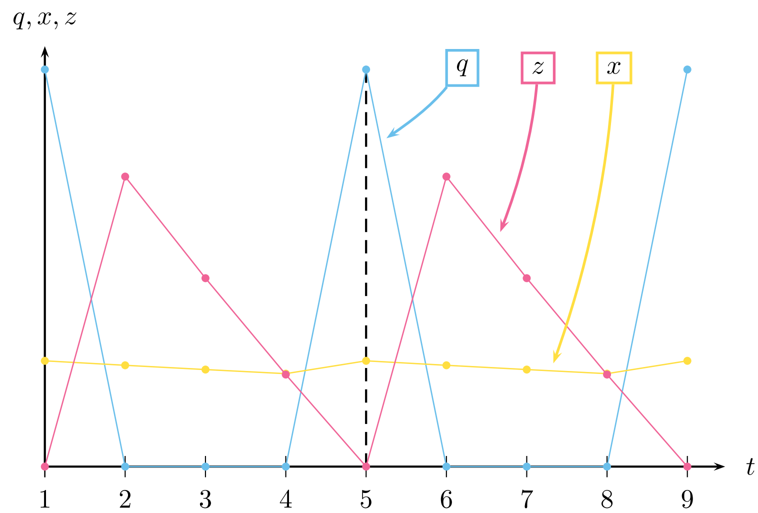

Figure 1: A simple money cycle. The symbols q, z and x denote production, real balances of money holdings, and consumption respectively.

A similar effect arises in Samuelson (1958) where consumption decreases over the life-cycle of an agent if the interest rate is away from zero. In Berentsen, Camera and Waller (2005), a money model based on the Lagos and Wright (2005) search and matching framework, the exogenous money cycle structure includes one period with a centralized market followed by two decentralized market periods. They find that trade is higher in the first of the decentralized markets, which is due to the hot potato effect. Our model shows that this effect is present even when money cycles are not imposed exogenously.

Smoothing out non-convexities with non-monotonic policies Agents in our model smooth out the non-convexity of the production cost function by following work-vacation patterns over time. This allows them to lower their average production cost below the cost of the average amount of production over the money cycle. However, unlike the models discussed below, production does not decrease monotonically throughout the money cycle.

There are three forces influencing agents' production decisions. First, agents must time their production so that they can finance their consumption in every period. Second, the fixed cost of production is minimized when production is concentrated in fewer periods. Third, the impact of the inflation tax is minimized when money is acquired immediately before it is spent. Unlike consumption, these forces do not give rise to monotonic production over the money cycle. Instead, agents may have several clusters of work days separated by vacations. Throughout the subsequence of production days, the quantity of production increases over the money cycle. Money holdings increase on work days and decrease on vacation days. When the marginal cost is constant, agents work once at the start of every money cycle, and money holdings decrease monotonically thereafter. However if the marginal cost is increasing, money holdings need not move monotonically throughout the money cycle.

Our model can be seen as a general equilibrium version of the models of Baumol (1952) and Tobin (1956). In these models, agents may pay a fixed cost to liquidate capital into money. Thus, the agents face a non-convex cost of acquiring cash. It is optimal for them to smooth out this cost by liquidating enough capital for several periods, and exhausting the resulting cash before replenishing. This means both money balances and liquidation decisions are monotonic. Money cycle equilibria, such as the one illustrated in Figure 1 may also have this property. However, if the marginal cost is increasing, money cycles may be more complicated.

In the model of Rogerson (1988), agents use lotteries to smooth out non-convexities in a indivisible labor supply choice and obtain an efficient equilibrium. The agents in Prescott, Rogerson and Wallenius (2009) and Rogerson and Wallenius (2009) smooth out a non-convex labor supply choice across time. Since agents have access to credit, the timing of labor-supply is irrelevant, and the equilibrium is efficient. If agents must settle trades immediately, the production timing becomes a non-trivial choice. Despite this complexity, our paper shows that under optimal monetary policy, agents may use money to smooth out non-convexities across time to obtain efficient equilibria.

Non-trivial distribution of money holdings Away from the Friedman rule, each agents' money holdings proceed through a finite money cycle which forms the support of the distribution of money holdings. Hence, at any point in time, there is an equal measure of agents at each point of the money cycle, e.g. in the example of Figure 1 this means that at any point in time there is one quarter of agents with no money holdings, one quarter of agents with money holdings at period two of the money cycle, etc. A non-trivial distribution of money naturally arises when agents either have heterogeneous preferences, or face idiosyncratic shocks.[1] Our model illustrates that fixed costs may also lead to non-trivial money distributions.

The paper is organized as follows: Section 2 presents the environment of our model. The money cycle equilibria are characterized in Section 3. The algorithm used for our numerical examples is presented in Section 4 and Section 5 concludes.

[1]Bhattacharya, Haslag and Martin (2005) supplies details of how exogenous heterogeneity leads to money distributions in several models.

2 Environment

We construct a stationary general equilibrium model with infinite discrete time where agents discount at rate β. In each period t, agents may produce any quantity $q_t \geq 0$ of the consumption good at cost $c(q_t)$. Agents receive utility $u(x_t)$ from consuming x_t units of this good. If an agent produces more than he consumes, he may sell the surplus for fiat money on a spot market at the price p_t.

As is standard in the monetary theory literature, we assume that agents must settle their transactions immediately. We interpret this as an assumption that agents are anonymous and have no way to enforce IOU's or private credit (such as a promise to repay in future). Let M_t be the aggregate stock of money in nominal terms at the beginning of period t. We assume that it evolves over time depending on a lump sum transfer T_t issued by the government to all agents before starting to trade goods. We write $1 + \pi_{t+1} = M_{t+1}/M_t$, where π is the money growth rate.

If an agent holds m units of money at time t and the continuation value of holding m is $W_t(m)$, his optimal choices satisfy the Bellman equation

$$\begin{aligned} W_t(m) = & \max_{q\in\mathbb{R}_+,\, x\in\mathbb{R}_+,\, m'\in\mathbb{R}_+} u(x) - c(q) + \beta\, W_{t+1}(m') \\ & \text{s.t.} \quad p_t\, x + (1+\pi_t)\, m' = p_t\, q + m + T_t. \end{aligned} \tag{1}$$

We focus on stationary equilibria under stationary monetary policy, so that the real transfer is stationary with $T = T_t/p_t$ and inflation is stationary with $\pi = \pi_t$. Let $Z_t = M_t/p_t$ be the real value of the money stock. Since $M_t + T_t = (1+\pi)\, M_t$, or in real terms, $Z_t + T = (1+\pi)\, Z_t$, the real money stock is stationary and can be expressed as $Z = T/\pi$. When we replace the nominal balances m_t with real balances $z_t = m_t/p_t$, the problem becomes stationary:

$$\begin{aligned} V(z) = & \max_{q\in\mathbb{R}_+,\, x\in\mathbb{R}_+,\, z'\in\mathbb{R}_+} u(x) - c(q) + \beta\, V(z') \\ & \text{s.t.} \quad x + (1+\pi)\, z' = q + z + T. \end{aligned} \tag{2}$$

Let us assume that the production cost $c(q)$ is strictly increasing, convex, and differentiable, but allow for a discontinuity at $q = 0$ which represents a fixed cost. We assume that $c(0) = 0$. Similarly, we assume that u is strictly increasing, differentiable, and concave and satisfies the Inada conditions.

We write the optimal production quantity as $q(z)$ and the optimal consumption policy as $x(z)$. The distribution of real money holdings is F. A *symmetric stationary equilibrium* in this environment is a tuple

$$[x(z), q(z), F(z), T]$$

such that

- the policies $q(z)$ and $x(z)$ solve the stationary problem above given T;
- goods and money markets clear so that supply equals demand:

$$\int q(z)\ dF(z) = \int x(z)\ dF(z) \quad \text{and} \quad \frac{T}{\pi} = \int z\, dF(z);$$

- the distribution of money holdings is stationary:

$$F(z') = \int I\left\{[q(z) - x(z) + z + T]/(1+\pi) \leq z'\right\}\ dF(z)$$

  which means that the measure of agents holding less than or equal to z' real balances of money at the beginning of the next period has to equal the measure that saved z' by working, consuming, and getting lump sum transfers.

3 Equilibrium

In this section, we characterize equilibria. Despite the fact that the value function is not differentiable, we show that the Euler equation holds when agents have non-zero money holdings. We establish that away from the Friedman rule, agents' decisions follow money cycles in every stationary equilibrium. In a money cycle, agents begin with no money holdings, work in the first period, and end the money cycle with no money again. We show that money can only have value for money cycles of length two or more. In a money cycle equilibrium, the distribution of money holdings is non-trivial, with a finite support. In money cycles, agents front-load consumption in response to the inflation tax. We interpret this as a hot potato effect of inflation. Agents also attempt to back-load production, but this is limited by the budget constraint and their preference to front-load consumption. In particular, money cycles begin with a work day and end with a vacation. When agents face a constant marginal production cost, money cycles have a monotonic Baumol-Tobin structure with only one work day. However, we provide an example with increasing marginal cost in which the money holdings are not monotonically decreasing throughout the money cycle. Finally, we study optimal monetary policy. We show that the Friedman rule is optimal. However, money cycle equilibria typically do not exist at the Friedman rule. Agents are able to perfectly smooth out the non-convexity from the fixed cost, but this typically requires a non-repeating work-vacation pattern.

The key equation we would like to establish for characterizing the equilibrium is the Euler equation

$$u'(x) = \frac{\beta}{1+\pi} u'(x'). \tag{3}$$

This equation is typically established by applying an envelope theorem. However, the fixed cost of production creates non-differentiabilities and non-convexities, which preclude the application of the traditional envelope theorem of Benveniste and Scheinkman (1979) and also the more recent envelope theorem of Cotter and Park (2006). We apply our Clausen and Strub (2010) envelope theorem to obtain the Euler equation without making any such assumptions. To apply our theorem, we rewrite the Bellman equation as

$$V(z) = \max_{z' \geq 0,\, q \geq 0} f(z, z', q) + \beta V(z')$$

where

$$f(z, z', q) = u[q + z + T - (1+\pi) z'] - c(q)$$

is differentiable in z and z'. Our envelope theorem then implies that when $z' > 0$,

$$-f_{z'}(z, z', q) = \beta V'(z') = \beta f_z(z', z'', q'),$$

where z'' and q' are optimal choices at z'. Expanding this equality gives the Euler equation (3). (Note that the Euler equation does not hold when $z' = 0$.)

The first-order condition

$$u'(x) = c'(q) \tag{4}$$

applies on work days (when $q > 0$).

Definition 1 We say that an agent's decisions $(\{q_t\}, \{x_t\}, \{z_t\})$ follow a *money cycle* of length n if n is the smallest number such that $z_t = z_{t+n}$ for all t. We say that the money cycle is *non-trivial* if $n > 1$. □

Theorem 1 *In every stationary monetary equilibrium away from the Friedman rule (i.e. for* $1 + \pi > \beta$*), agents' decisions follow a (possibly trivial) money cycle that contains* 0*.* □

Proof Suppose $(\{q_t^*\}, \{x_t^*\}, \{z_t^*\})$ is an optimal solution to the agent's problem. We argue below that z_t^* includes 0 for some t. By truncating the start of the sequences, we repeat the argument to conclude that z_t^* includes a second 0. In a stationary equilibrium, the same decisions are taken both times $z_t^* = 0$. We conclude that the entire sequence of decisions between the first and second time $z_t^* = 0$ are repeated over and over to form a money cycle.

For the sake of contradiction, suppose that $z_t^* > 0$ for all t. In this case, the Euler equation (3) applies every period so that

$$u'\left(x_t^*\right) = \left(\frac{\beta}{1+\pi}\right)^n u'\left(x_{t+n}^*\right). \tag{5}$$

When the money growth rate is above the Friedman rule (i.e. $1+\pi > \beta$), the first term on the right side converges to 0, so $u'\left(x_t^*\right) \to \infty$ and hence $x_t^* \to 0$. Similarly, the first-order condition (4) that $u'\left(x_t^*\right) = c'\left(q_t^*\right)$ combined with the fact that there is no last work date (due to market clearing) implies that a subsequence of q_t^* diverges to ∞ with marginal cost also diverging to ∞. We now show these decisions are suboptimal in a stationary monetary equilibrium, and conclude that z_t^* contains 0. We provide separate proofs for the inflation, deflation and boundary cases.

Under inflation, $T > 0$ implies that consuming $x_t = T$ every period is feasible (without relying on savings). When consumption drops below T, an agent would be strictly better of deviating to consuming T every period.

Under deflation, the transfers T are negative, so they can not be used to fund consumption. But real money holdings z give a real return of $z\,(-\pi)/(1+\pi)$. When the real balance z is sufficiently large, the real return will be bigger than the lump sum tax $-T$, and the remainder, $x\,(z)$ can be consumed forever. We already showed that the real balances raised on work periods approaches ∞. This means the agent can afford $x\left(z_t^*\right) \to \infty$ units of consumption forever. Therefore $x_t^* \to 0$ is suboptimal.

Finally, in the boundary case when $T = 0$ and $\pi = 0$, the budget constraint simplifies to

$$x_t + z_{t+1} = q_t + z_t.$$

This budget constraint allows the agent to perfectly store the consumption good by holding money. Since production q_t^* increases towards infinity, and consumption x_t^* decreases towards 0, eventually production on work days is larger than consumption. Therefore, a sequence of multiple work days followed by one vacation is suboptimal: the agent should swap the first work day and the vacation day, so that the production cost is delayed. This implies that eventually there are no consecutive work days. Moreover, since marginal cost is unbounded (from the Euler equation and first-order condition), there is a time t where[2]

$$c\left(q_t^*\right) > 2\,c\left(q_t^*/2\right).$$

This cannot be optimal as an agent is strictly better off splitting their production across two consecutive days (which is feasible because of the budget constraint). ■

[2] To see this, recall a function f is convex if $f\left[\alpha\,x + (1-\alpha)\,y\right] \leq \alpha\,f\,(x) + (1-\alpha)\,f\,(y)$. Setting $\alpha = 0$ and $x = 0.5$ gives $2\,f\,(y/2) - f\,(0) \leq f\,(y)$. By taking the convex hull of $c\,(q)$, we arrive at a convex function $f\,(q)$ with $f\,(0) = 0$ that coincides with c for sufficiently large q.

Note that there may be multiple stationary equilibria. In this case, it would also be an equilibrium for agents to switch from one money cycle to another. However, such an equilibrium is not stationary.

The following corollary shows that a trivial money cycle (in which agents always hold zero real-balances of money) can not be a monetary equilibrium. Rather, trivial money cycles give rise to autarky.

Corollary 1 *In every trivial money cycle equilibrium (with length one), money has no value.* □

Proof As shown above, every money cycle includes zero. Together with stationarity this implies that all agents consume what they produce. Thus the money market clearing condition implies:

$$\frac{M_t}{P_t} = \int z_t \, dF\left(m_t\right) = 0$$

It follows that $p_t = \infty$ and thus there exists no monetary equilibrium. ■

Next, we show that the distribution of money holdings is non-degenerate, but has a simple structure. This is because agents trade positions with each other in the money cycle.

Corollary 2 *The stationary distribution F of (real balances of) money holdings has equal mass over a finite set.* □

Proof The support of the distribution of real balances coincides with the equilibrium sequence of real balances. Since each agent cycles through the sequence at the same pace, the measure of agents at each point of the sequence is equal, so the stationary distribution has equal mass at each point in its support. ■

Without loss of generality, we say that the start of the money cycle is when agents hold no money. The following theorem summarizes the properties of money cycles.

Theorem 2 *In every stationary equilibrium away from the Friedman rule, agents proceed through money cycles that*

(i) have decreasing consumption, with marginal utility increasing in proportion to the inflation tax, $\left(1+\pi\right)/\beta$.

(ii) after eliminating vacations have production increasing throughout the money cycle, with (shadow) marginal cost increasing in proportion to the inflation tax.

(iii) begin with work and end with vacation. □

Proof (i) Previously, we found the Euler equation (3) holds between period t and $t+1$ whenever $z_{t+1}^{*} > 0$. Since z_{t+1}^{*} is greater than 0 in every period before the end of a money cycle, the Euler equation holds between every period within a money cycle. The Euler equation implies consumption decreases with marginal utility increasing in proportion to $(1+\pi)/\beta$.

(ii) Follows from part (i) and the production first-order condition (4).

(iii) Since the agent begins a money cycle with no money, it must work to finance its first period consumption.

Suppose the agent works in the last period. Since production is greater or equal to consumption in the first period, parts (i) and (ii) imply that production is strictly greater than consumption in the last period. This contradicts the conclusion that savings are 0 in the last period. ■

The following corollary shows that when agents face a constant marginal production cost, a Baumol-Tobin style work-vacation pattern such as the example in Figure 1 emerges endogenously.

Corollary 3 *If $c(q)$ is affine away from $q=0$, then money cycles contain only one work day.* □

Proof Since $c'(q_t) = u'(x_t)$ on work days, and $u'(x_t)$ increases over the money cycle, it follows that $c'(q_t)$ must increase on work days. But when $c(q)$ is affine, $c'(q)$ is a constant. ■

However, if the marginal cost is strictly increasing, money cycles can become more complex. Figure 2 shows a non-trivial example of a money cycle including the resulting distribution of money holdings. Agents proceed through a money cycle of five time periods with vacations on the third and final periods. Consumption x is decreasing over the money cycle whereas the production quantity q is increasing, conditional on producing. Money holdings z are non-zero (even at $t=4$!) until the money cycle ends with zero money holdings. Moreover, consumption and production are not monotonic in money holdings; i.e. agents with more money do not work more, and do not consume more. The histogram of real balances presents a non-degenerate distribution of money holdings. Goods as well as money markets clear in each period of the money cycle. Hence, aggregate consumption and production are stationary over time.

Let us finally examine the case of the Friedman rule. The following theorem shows that under the Friedman rule, agents can perfectly smooth out the non-convexity of the cost function. To be precise, we smooth out the agent's cost function $c(q)$ by defining its convex hull $\bar{c}(q)$ as the upper envelope of the set of affine functions that lie below $c(q)$. The convex

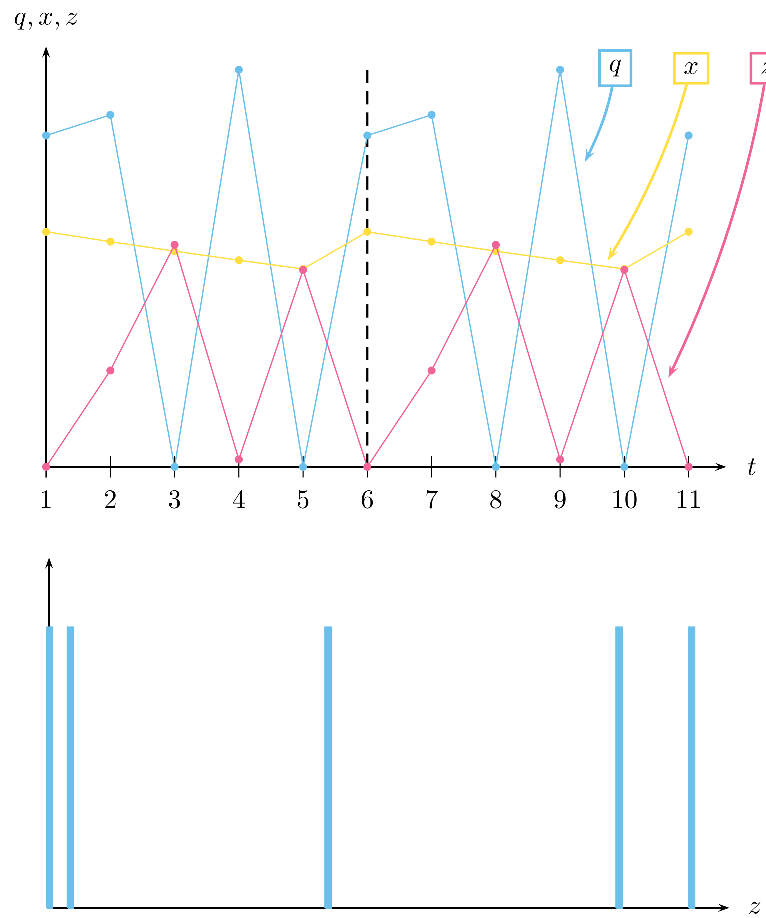

Figure 2: Equilibrium money cycle and related histogram of money holdings when $u(x) = x^{0.3}$, $c(q) = 0.1 + q^{1.5}$ for $q > 0$, $\beta = 0.98$ and $\pi = 0.01$.

hull is depicted in Figure 3. We say that an agent can smooth out the non-convexity of $c(q)$ if it can achieve the same utility under $c(q)$ as if it instead faced $\bar{c}(q)$. We show below that this is possible under the Friedman rule. However, smoothing out non-convexities may not be possible with a money cycle, which requires a rational ratio of work days to vacation days. Therefore, the set of primitives for which money cycles exist at the Friedman rule has measure zero. This contrasts with Theorem 1 which shows that money cycles exist away from the Friedman rule.

Theorem 3 *Consider the monetary policy at the Friedman rule with* $1 + \pi = \beta$. *Suppose the* *convex hull $\bar{c}(q)$ coincides with $c(q)$ for $q \geq \bar{q}$.*[3] *Let $x^{**} = q^{**}$ be the efficient symmetric steady state in the convexified economy. Figure 3 illustrates $c(q)$, $\bar{c}(q)$, q^{**} and $\bar{q}$.*

(i) Every equilibrium gives every agent the same value as in the efficient symmetric steady state of the convexified economy. Hence agents may smooth out non-convexities, and the Friedman rule is optimal.

(ii) A money cycle equilibrium exists if and only if $q^{**}/\bar{q} \in \mathbb{Q}$ or $q^{**} > \bar{q}$, □

Proof When $q^{**} \geq \bar{q}$, the equilibria in the convexified and the fixed cost economies coincide, and these results are trivial. For the remainder of the proof, we consider the case $q^{**} < \bar{q}$.

(i) The steady state in the convexified economy is efficient and symmetric. This gives agents a value of $[u(x^{**}) - c(x^{**})] / (1-\beta)$. Now consider the following sequence in the economy with fixed cost: each agent consumes x^{**} and on each production day produces $\bar{q}$. In the long run, the fraction of times each agent works approaches $q^{**}/\bar{q}$. (For example, whether an agent works at time t could be determined by whether the t^{th} digit of the binary expansion of $q^{**}/\bar{q}$ is 1.) Such an allocation is optimal for all agents since they are indifferent about the timing decision of production at the Friedman rule. This can be seen from the Euler equation $u'(x_t) = u'(x_{t+n})$ and the first-order condition $u'(x_t) = c'(q_t)$. Since this gives each agent the same value as in the convexified economy, it is efficient.

(ii) If $q^{**}/\bar{q} \in \mathbb{Q}$ with $q^{**}/\bar{q} = a/b$, then each agent can work a days and take $b-a$ days off, which gives rise to a money cycle equilibrium. However, any cycle gives a rational ratio of work days to vacation days, so the average number of work days can not approach an irrational number $q^{**}/\bar{q} \notin \mathbb{Q}$. ■

4 Algorithm

This section describes the algorithm that we use to calculate the examples in Figures 1 and 2.[4] The algorithm exploits the fact that it is easy to calculate the agent's optimal plan once the starting consumption value is known. Therefore we use a forward shooting algorithm

[3] Note that when $c(q)$ has constant marginal cost, the convex hull $\bar{c}(q)$ does not intersect with $c(q)$ for $q > 0$.

[4] The program code is available on the authors' websites.

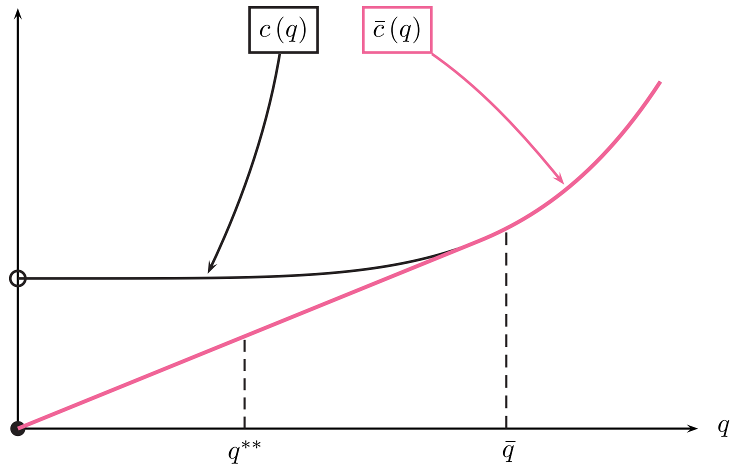

Figure 3: The convex hull of the cost function

that searches among possible consumption values at the start of the money cycle.[5] We then calculate the equilibrium by searching for the real value of monetary transfers that clears the market. (This is equivalent to deriving the real value of one unit of money.)

Even though we established that the length of a money cycle is finite, we did not yet provide a bound for the length of money cycles. This is important, as our algorithm calculates the optimal choices for each candidate money cycle length, and can not consider an infinite number of candidates. The following theorem provides a bound on the length of money cycles when there is inflation, and also bounds the possible initial consumption values x_1.

Theorem 4 *Consider an equilibrium with inflation so that $T > 0$.*

(i) If the agent holds real balances of z_1, then

(a) their consumption x_1 lies in $[T, \bar{x}_1]$, where $\bar{x}_1 = \max\{z_1 + T, \hat{x}\}$ and $\hat{x}$ solves $u'(x) = c'(x - z_1 - T)$.

(b) they spend all of their money within the following number of periods,

$$\left\lceil \log \frac{u'(\bar{x}_1)}{u'(T)} \Big/ \log \frac{\beta}{1+\pi} \right\rceil .$$

[5] Forward shooting methods are the topic of Judd (1998, Chapter 10).

(ii) Since money cycles begin with $z_1 = 0$, the length of money cycles is bounded by this expression at $z_1 = 0$.

Proof (i) (a) Clearly $x_1 \geq T$. We need to show that $x_1 \leq \max\{z_1 + T, \hat{x}\}$. If the agent does not work in the first period, then $x_1 \leq z_1 + T$. If the agent works, we will show that $x_1 \leq \hat{x}$. Intuitively, if an agent consumes a lot, then they must also produce a lot; but as diminishing marginal utility and increasing marginal cost set in, it becomes suboptimal to increase consumption and production. Since the agent starts with z_1 real balances of money, the budget constraint implies that $q_1 \geq x_1 - z_1 - T$. The first-order conditions imply $u'(x_1) = c'(q_1)$. Moreover, since marginal cost is increasing, $c'(q_1) \geq c'(x_1 - z_1 - T)$. Thus, $u'(x_1) \geq c'(x_1 - z_1 - T)$, or equivalently, $x_1 \leq \hat{x}$ since u' is decreasing and c' is increasing.

(b) Now suppose that $z_2, \ldots, z_n > 0$. We will put an upper bound on n for which this can be true. Under inflation, $x_n \geq T > 0$. By the Euler equation,

$$u'(x_1) = \left(\frac{\beta}{1+\pi}\right)^n u'(x_n).$$

Substituting the bound for x_1 above and the bound $x_n \geq T$ into this equation gives

$$u'[\max\{z_1 + T, \bar{x}_1\}] \leq u'(x_1) = \left(\frac{\beta}{1+\pi}\right)^n u'(x_n) \leq \left(\frac{\beta}{1+\pi}\right)^n u'(T),$$

which can be rearranged to the bound on n given above.

(ii) Trivial. ■

The algorithm takes u, c, β and π as exogenous, and calculates the money cycle length n, the real value of transfers T and the decisions $(x_t, q_t, z_t)_{t=1}^n$.

Algorithm to solve the agent's problem: Given $T > 0$, calculate the length of the money cycle n, and the decisions $(x_t, q_t, z_t)_{t=1}^n$.

(i) Calculate bounds on initial consumption x_1 and money cycle length n according to Theorem 4.

(ii) Search in this range for the x_1 that maximizes lifetime utility. This utility is calculated as follows. For each possible money cycle length (within the bounds from above), calculate the sequence of (x_t, q_t, z_t):

(a) Each x_t is calculated using the Euler equation.

(b) Set $q_t = 0$ if z_t is enough to finance x_t. Otherwise, $c'(q_t) = u'(x_t)$.

(c) Set $z_1 = 0$, and the remaining z_t are determined by the budget constraint. (Note: until we find the best x_1, there maybe money left over at the end of the money cycle.)

(d) The value of this sequence is then $\frac{1}{1-\beta^n} \sum_{t=1}^{n} \beta^t \left[u(x_t) - c(q_t)\right]$.

Algorithm to calculate the money cycle equilibrium: Find the $T > 0$ such that markets clear using the bisection algorithm[6] on the excess demand, which is $\sum_{t=1}^{n} x_t - q_t$ where n, x_t and q_t are the optimal decisions calculated in the inner loop.

5 Conclusion

We present a simple general equilibrium model with homogeneous agents where money arises from a fixed cost of production. This leads to a rich equilibrium structure. Agents avoid the fixed cost by taking vacations. Fiat money then supports the trade of surplus production between workers and vacationers. Away from the Friedman rule, we show that agents acquire and spend money in money cycles of finite length. Throughout a money cycle, agents decrease their consumption which we interpret as the hot potato effect of inflation. The Friedman rule is optimal and we show that a money cycle may still exist.

Classical models of money are typically based on a competitive market with neither capital nor credit. They then impose exogenous timing structures, market participation constraints, or cash-in-advance constraints to make money essential. Our model provides a novel reason for fiat money to be essential: a fixed cost of production. The equilibria that arise in our simple model are similar to other classical money models: the Friedman rule is optimal, and monetary equilibria Pareto dominate the non-monetary equilibrium. Our model can be considered a general equilibrium version of the models of Baumol (1952) and Tobin (1956). In their models, a constant marginal cost of liquidation leads the agent to proceed through a monotonic cycle of liquidating and spending money balances down to zero. However, in our model, more complex money cycles may arise if the marginal cost is increasing. We calculate an example where money holdings do not decrease monotonically throughout the money cycle using a forward shooting algorithm. Moreover, our model yields a non-degenerate distribution of money holdings in the absence of idiosyncratic shocks or any other type of heterogeneity among agents.

[6]The bisection algorithm calculates a root of a continuous function $f : [a, A] \to \mathbb{R}$ with $f(a) < 0 < f(A)$ by successively cutting the domain $[a, A]$ in half around the mid-point $(a + A)/2$. See Judd (1998, page 148).

There are natural extensions of interest. We do not study uncertainties in our framework. Shocks on productivity or on the fixed cost of production will certainly have a lasting effect on money cycle equilibria. Such uncertainties may also call for alternative, state dependent monetary policy transmission mechanisms to examine further.

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