Technical Appendix for the Paper
"The Dynamics of Government"
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In this appendix, we provide a complete set of formal proofs of the Propositions of the paper "The Dynamics of Government" (*Journal of Monetary Economics*, October 2005) – excluding Proposition 1 which is proved in the text –. For each Proposition, we provide an unabridged restatement before the respective proof. The numbering of equations follow that of the paper.

Proposition 2 *For $a = 1$ and any $\omega \in [0, 1]$, the political equilibrium is characterized as follows:*

$$\begin{aligned} B^n(u_t) &= \begin{cases} \frac{\rho}{w}(u_t - u^n) & \text{if } u_t \geq u^n \\ 0 & \text{else} \end{cases} \\ U^n(b_t) &= u^n + \frac{w}{2-\beta\rho} b_t, \end{aligned}$$

where

$$\rho = \frac{2Z}{1+\beta Z}$$

and $Z \in [0, 1/2]$ is the real solution to

$$Z\left(1 + \omega\beta Z^2\right) = \frac{1-\omega}{2},$$

and is decreasing in ω and β. Given any $u_0 > u^n$, the equilibrium law of motion is

$$u_{t+1} = u^n + Z(u_t - u^n).$$

The economy converges monotonically to a unique steady state with $b = b^n = 0$ and $u = u^n$. For $u_0 \leq u^n$, $u_t = u^n$ $\forall t > 0$. Finally, ρ is decreasing in ω and β; if $\omega = 1$, then, $\rho = 0$, implying immediate convergence to the steady state.

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Proof. When $R=0$, and for $\omega\in[0,1]$, the political objective function can be written as

$$\begin{aligned}
V &= \mu\left(1-u_t\right)w+\mu u_t b_t w+\left(1-\mu\right)b_t w\\
&\quad +\omega\left(1-\mu\right)\left(b_t+\beta b_{t+1}\right)w\\
&\quad +\omega\mu e_t\left(1+\beta\right)w+\omega\mu\left(1-e_t\right)\left(b_t+\beta b_{t+1}\right)w\\
&\quad -\omega\mu\left(e_t\right)^2\\
&\quad -\left(1+\omega\right)\tau_t-\omega\beta\tau_{t+1}.
\end{aligned}$$

Here, it is convenient to guess that $b_t=\alpha_1\left(u_t-u^n\right)$ for a coefficient α_1 yet to be determined. We first solve for $U\left(b_t\right)$ to obtain

$$U\left(b_t\right)=u^n+\frac{w}{2-\beta\alpha_1 w}b_t,$$

delivering $B\left(U\left(b_t\right)\right)=\frac{w\alpha_1}{2-\beta\alpha_1 w}b_t$. Defining $Z=\frac{\alpha_1 w}{2-\beta\alpha_1 w}$, this implies $b_{t+1}=Zb_t$ and $b_{t+2}=Z^2b_t$ and

$$\begin{aligned}
e_t &= \left(1+\beta-\left(1+\beta Z\right)b_t\right)\frac{w}{2}\\
\tau_t &= \left(1-\frac{\mu}{2}\left(1-u_t+\left(1+\beta-\left(1+\beta Z\right)b_t\right)\frac{w}{2}\right)\right)b_t w\\
\tau_{t+1} &= \left(1-\frac{\mu w}{4}\left(2\left(1+\beta\right)-\left(1+Z\left(1+\beta+\beta Z\right)\right)b_t\right)\right)Zb_t w.
\end{aligned}$$

We now note that

$$\begin{aligned}
\frac{db_{t+1}}{db_t} &= Z,\ \frac{db_{t+2}}{db_t}=Z^2,\ \frac{de_t}{db_t}=-\left(1+\beta Z\right)\frac{w}{2}\\
\frac{d\tau_t}{db_t} &\equiv T_0=\left(2-\mu\left(1-u_t-w\left(1+\beta Z\right)b_t+\frac{w\left(1+\beta\right)}{2}\right)\right)\frac{w}{2}\\
\frac{d\tau_{t+1}}{db_t} &\equiv T_1=\frac{1}{2}\mu w^2 Z\left(1+Z\right)\left(1+\beta Z\right)b_t+wZ-\mu w\frac{1}{2}w\left(1+\beta\right)Z.
\end{aligned}$$

The first-order condition for maximizing the political objective is then

$$0=\mu u_t w+\left(1+\omega\left(1+\beta Z\right)\right)\left(1-\mu\right)w+\omega\mu\left(1-e_t\right)\left(1+\beta Z\right)w-\left(1+\omega\right)T_0-\omega\beta T_1$$

yielding

$$b_t=\frac{1}{w}\frac{1-\omega}{1+Z\left(\beta+\omega\beta Z\left(1+\beta Z\right)\right)}\left(u_t-u^n\right)$$

verifying the guess, provided

$$\alpha_1=\frac{1}{w}\frac{1-\omega}{1+Z\left(\beta+\omega\beta Z\left(1+\beta Z\right)\right)}.$$

From the definition of Z we obtain $\alpha_1=\frac{2Z}{(1+\beta Z)w}$. Thus,

$$\frac{2Z}{\left(1+\beta Z\right)w}=\frac{1}{w}\frac{1-\omega}{1+Z\left(\beta+\omega\beta Z\left(1+\beta Z\right)\right)}, \tag{23}$$

which can be rewritten

$$\left(\omega\beta Z^2+1\right)Z=\frac{1-\omega}{2}.$$

The left-hand side is monotonically increasing from zero as Z increases from zero. Furthermore, an increase in ω increases the left-hand side while it reduces the right-hand side. The solution in terms of Z therefore falls. Similarly, an increase in β increases the left-hand side, provided $\omega > 0$, implying that that solution in terms of Z falls, unless it is zero. Since α_1 is monotonically increasing in Z, we have established that α_1 is decreasing in ω and β. From this it also follows that the constraint $B(u) \in [0, 1]$ is satisfied for any $u \geq u^n$ and we set $B(u) = 0$ for $u < u^n$, which, as in the previous proof, neither affects the objective function nor $U(b_t)$. ■

Proposition 3 *Assume $\omega = 0$ and risk aversion ("a"). Then if $R \in [0, R_{\max}]$, where $R_{\max} > 1$ is defined below, the political equilibrium is characterized as follows:*

$$\begin{aligned} B^{ao}(u_t) &= \begin{cases} b^{ao} + \frac{\rho}{w}(u_t - u^{ao}) & \text{if } u_t \geq u^{ao} - \frac{w}{\rho} b^{ao} \\ 0 & \text{otherwise} \end{cases} \\ U^{ao}(b_t) &= \underline{u}^{ao} + \frac{w}{2 - \beta\rho}(b_t - b^{ao}), \end{aligned}$$

where

$$\begin{aligned} \rho &= \frac{2Z}{1 + \beta Z} \\ Z &= \frac{1}{2}\frac{1 - R}{1 + R} \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ b^{ao} &= \frac{4R}{1 + 3R}\frac{1 + \beta}{2 + \beta}, \\ u^{ao} &= 1 - \frac{w}{2}(1 + \beta)(1 - b^{ao}), \end{aligned}$$

and $R_{\max}$ is defined as the R such that $B^{ao}(0) = 1$. If $R > 0$, then $b^{ao} > 0$ and $b_t > 0 \ \forall \ t > 0$ (redistribution is positive after at most one period). Furthermore, b^{ao} and u^{ao} increase in R. For $t > 0$, the equilibrium law of motion is

$$u_{t+1} = u^{ao} + Z(u_t - u^{ao}).$$

Given u_0, the economy converges to a unique steady state with $b = b^{ao}$ and $u = u^{ao}$ following an oscillating (monotone) path if $R > (<) 1$. If $R = 1$, convergence is immediate.

Proof. As above, guess $B(u_t) = \alpha_0 + \alpha_1 u_t$. Then

$$B(U(b_t)) = \alpha_0 + \alpha_1 \frac{2 - (1 + \beta(1 - \alpha_0)) w}{2 - \beta\alpha_1 w} + b_t \frac{\alpha_1 w}{2 - \beta\alpha_1 w}. \tag{24}$$

Using this in the first-order condition for maximizing (5) over b_t and solving for b_t yields

$$b_t = \frac{(2 - \alpha_1 w\beta) R}{w(1 + R)} - \frac{1}{w} + \frac{1 + \beta(1 - \alpha_0)}{2} + \frac{1}{2}\frac{(1 - R)(2 - \beta\alpha_1 w)}{(1 + R) w} u_t,$$

which is linear, as conjectured. We thus need to set

$$\begin{aligned} \alpha_0 &= -\frac{2 - w(1 + \beta)}{w(2(1 + R) + \beta(1 - R))} \\ &\quad + R\frac{(\beta(1 - \beta) + 2) w + 8 - 2(2 - \beta)}{(2(1 + R) + \beta(1 - R)) w (2 + \beta)} \\ \alpha_1 &= \frac{2(1 - R)}{w(2 + \beta + R(2 - \beta))} = \frac{\rho}{w}. \end{aligned}$$

Noting that $B(U(b_t))$ and $U(B(u))$ have a fixed point, respectively, at

$$\begin{aligned} b^{ao} &= R\frac{4(1+\beta)}{(2+\beta)(1+3R)}, \\ u^{ao} &= 1-\frac{w}{2}(1+\beta)(1-b^{ao}) \end{aligned}$$

we can write

$$\begin{aligned} B(u_t) &= \alpha_0+\alpha_1 u_t \\ &= b^{ao}+\frac{\rho}{w}(u_t-u^{ao}). \end{aligned}$$

Next we need to prove that $\exists R_{\max}>1$ such that if $R<R_{\max}$ then the constraint $b\leq 1$ is never binding in equilibrium. Also, we show that $R_{\max}$ is such that $B(0;R_{\max})=1$. First, since $B(u)$ is linear, its maximum is attained at either $u=0$ or $u=1$. Hence, if $1\geq\phi(R)=\max\{B(0;R),B(1;R)\}$, the constraint does not bind. Second, standard algebra establishes that $\phi'(R)>0$ since $\frac{\partial B(0;R)}{\partial R}>0$ and $\frac{\partial B(1;R)}{\partial R}>0$. Hence, $\phi(.)$ is a one-to-one mapping and admits use of the inverse function $\phi^{-1}(.)$, and there must exist a unique $R_{\max}$ such that $\phi(R_{\max})=1$. Third, $R_{\max}$ is larger than one, since $\phi(1)=\frac{1+\beta}{2+\beta}<1$. Fourth, we know that, when $R>1$, $B(0;R)>B(1;R)$. Thus, $R_{\max}$ must be such that $\phi(R_{\max})=B(0;R_{\max})=1$.

To prove that benefits are strictly positive after at most one period, observe that, in equilibrium, the law of motion $b_{t+1}=B^{ao}(U^{ao}(b_t))$, conditioned on $u_{t+1}\geq u^{ao}-\frac{w}{\rho}b^{ao}$, can be written

$$\begin{aligned} b_{t+1} &= b^{ao}+\frac{1}{2}\frac{1-R}{1+R}(b_t-b^{ao}) \\ &= b^{ao}\frac{1}{2}\frac{1+3R}{1+R}+\frac{1}{2}\frac{1-R}{1+R}b_t. \end{aligned}$$

Clearly, whenever $\frac{1}{2}\frac{1-R}{1+R}>0$, $b_t\geq 0$ implies $b_{t+1}\geq 0$, since $b^{ao}\geq 0$, implying that $u_{t+1}\geq u^{ao}-\frac{w}{\rho}b^{ao}$. Consider the other case, i.e., consider $\frac{1}{2}\frac{1-R}{1+R}\leq 0$. Then, since $b_t\leq 1$, we have

$$\begin{aligned} b_{t+1} &\geq b^{ao}\frac{1}{2}\frac{1+3R}{1+R}+\frac{1}{2}\frac{1-R}{1+R} \\ &= \frac{1}{2}\frac{R(2+3\beta)}{(2+\beta)(1+R)}+\frac{1}{2(1+R)}>0. \end{aligned}$$

Whenever $b_t\geq 0$, then $b_{t+1}>0$ for any $t\geq 0$. Since we assume that u_0 is arbitrary, we cannot rule out that the constraint that $b_0\geq 0$ binds. ■

Proposition 4 *Assume* $0\leq R\leq R_{\max}$ *and* $\omega\in[0,1]$. *The political equilibrium is then characterized as follows:*

$$\begin{aligned} B^a(u_t) &= \begin{cases} b^a+\frac{\rho}{w}(u_t-\underline{u}^a) & \text{if } u_t\geq \underline{u}^a-\frac{w}{\rho}b^a \\ 0 & \text{otherwise} \end{cases} \\ U^a(b_t) &= \underline{u}^a+\frac{w}{2-\beta\rho}(b_t-b^a), \end{aligned}$$

where

$$\rho=\frac{2Z}{1+\beta Z}$$

and Z is a constant with the following properties: (i) $Z \in (-4/7, 1/2]$, (ii) $dZ/d\omega < 0$, (iii) $Z > (<) \, 0$ iff $R < (>) \frac{1-\omega}{1+\omega}$; and (iv) if $Z > 0$, $dZ/dR < 0$.
For $t \geq 0$, the equilibrium law of motion is

$$u_{t+1} = \underline{u}^a + Z \left(u_t - \underline{u}^a\right),$$

where $\underline{u}^a$ and b^a are functions of Z defined in the appendix. Given any u_0, the economy converges to a unique steady state following an oscillating (monotone) path if $R > (<) \frac{1-\omega}{1+\omega}$. If $R = \frac{1-\omega}{1+\omega}$, convergence is immediate to a steady state where $b = R\left(1+\omega\right)^2 / \left[\left(2+\beta\right)/\left(1+\beta\right) + \omega\left(1-\omega\right)\right] \geq 0$.

Proof. The constant Z is here defined as the unique real solution to the cubic equation

$$Z\left(\left(1+\omega\beta Z^2\right) + R\left(1+\omega\left(1+\beta Z\left(1+Z\right)\right)\right)\right) = \frac{1+\omega}{2}\left(\frac{1-\omega}{1+\omega} - R\right),$$

as the following argument shows. With $\omega \geq 0$, the political objective function can be written as:

$$\begin{aligned} V &= \mu\left(1-u_t\right)w + \mu u_t b_t w + \left(1-\mu\right)b_t w + R b_t w \\ &\quad + \omega\left(1-\mu\right)\left(b_t + \beta b_{t+1}\right)w + \omega R b_t w \\ &\quad + \omega\mu e_t\left(1+\beta\right)w + \omega\mu\left(1-e_t\right)\left(b_t + \beta b_{t+1}\right)w \\ &\quad - \omega\mu\left(e_t\right)^2 \\ &\quad - \left(1+\omega\right)\left(1+R\right)\tau_t - \omega\left(1+R\right)\beta\tau_{t+1}. \end{aligned}$$

The usual guess, $b_t = a_0 + \alpha_1 u_t$, substituted into equilibrium condition 2 yields (24), which we write $b_{t+1} = X + Z b_t$, where $\alpha_0 + \alpha_1 \frac{2-\left(1+\beta\left(1-\alpha_0\right)\right)w}{2-\beta\alpha_1 w} \equiv X$ and $\frac{\alpha_1 w}{2-\beta\alpha_1 w} \equiv Z$, which implies $b_{t+2} = X\left(1+Z\right) + Z^2 b_t$. Furthermore, we note that

$$\begin{aligned} \frac{db_{t+1}}{db_t} &= Z, \frac{db_{t+2}}{db_t} = Z^2, \frac{de_t}{db_t} = -\left(1+\beta Z\right)\frac{w}{2} \\ \frac{d\tau_t}{db_t} &= w\left(1-\mu\right) + \frac{1}{2}w\mu u_t \\ &\quad + \frac{1}{2}w\mu\left(1 + w\left(1+\beta Z\right)b_t - \frac{w\left(1+\beta\left(1-X\right)\right)}{2}\right) \\ \frac{d\tau_{t+1}}{db_t} &= \frac{1}{2}\mu w^2 Z\left(1+Z\right)\left(1+\beta Z\right)b_t + wZ\left(1-\mu\right) \\ &\quad + \mu w\left(1 - \frac{1}{2}w\left(1+\beta\right)\right)Z + \left(1+Z\left(2+\beta\left(3+2Z\right)\right)\right)\frac{1}{4}\mu w^2 X. \end{aligned} \tag{25}$$

The first-order condition for maximizing b_t can then be written

$$\begin{aligned} 0 &= \mu u_t w + \left(1+\omega\right)\left(1+R-\mu\right)w + \omega\left(1-\mu\right)\beta Z w \\ &\quad + \omega\mu\left(1-e_t\right)\left(1+\beta Z\right)w - \left(1+\omega\right)\left(1+R\right)\frac{d\tau_t}{db_t} - \omega\left(1+R\right)\beta\frac{d\tau_{t+1}}{db_t}, \end{aligned}$$

implying

$$b_t = \alpha_1 u_t + \alpha_0,$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{1}{w}\frac{(1+\omega)\left(\frac{1-\omega}{1+\omega}-R\right)}{1+Z\left(\beta+\omega\beta Z\left(1+\beta Z\right)\right)+R\left(1+\omega+\beta Z\left(1+\omega\left(2+Z\left(1+\beta\left(1+Z\right)\right)\right)\right)\right)}, \qquad (26)\\
\alpha_0 &= \alpha_{00}X+\alpha_{01},\\
\alpha_{00} &= -\frac{\beta}{2}\frac{1+\omega Z\left(2+\beta\left(1+2Z\right)\right)+\left(1+\omega\left(2+Z\left(2+\beta\left(3+2Z\right)\right)\right)\right)R}{1+Z\left(\beta+\omega\beta Z\left(1+\beta Z\right)\right)+R\left(1+\omega+\beta Z\left(1+\omega\left(2+Z\left(1+\beta\left(1+Z\right)\right)\right)\right)\right)},\\
\alpha_{01} &= -\frac{1}{2w}\frac{\left(4Z\beta\omega-\left(1+\omega\right)\mu 2-w\left(1+\beta\right)\mu\left(1+\omega+2Z\beta\omega\right)\right)R}{\mu\left(1+Z\left(\beta+\omega\beta Z\left(1+\beta Z\right)\right)+R\left(1+\omega+\beta Z\left(1+\omega\left(2+Z\left(1+\beta\left(1+Z\right)\right)\right)\right)\right)\right)}\\
&\cdot\frac{\left(\left(\left(\mu\beta\left(-w\left(1+\beta\right)\right)+2\beta\right)2Z\right)\omega-\left(1+\omega\right)\left(\mu w\left(1+\beta\right)+2\mu\right)\right)R}{\mu\left(1+Z\left(\beta+\omega\beta Z\left(1+\beta Z\right)\right)+R\left(1+\omega+\beta Z\left(1+\omega\left(2+Z\left(1+\beta\left(1+Z\right)\right)\right)\right)\right)\right)}\\
&-\frac{1}{2w}\frac{\mu\left(1-\omega\right)\left(2-w\left(1+\beta\right)\right)}{\mu\left(1+Z\left(\beta+\omega\beta Z\left(1+\beta Z\right)\right)+R\left(1+\omega+\beta Z\left(1+\omega\left(2+Z\left(1+\beta\left(1+Z\right)\right)\right)\right)\right)\right)}.
\end{aligned}
$$

Rewriting the expression for α_1 using the definition of Z yields

$$
Q^a\left(Z;R,\omega,\beta\right)\equiv 2Z\left(\left(1+\omega\beta Z^2\right)+R\left(1+\omega\left(1+\beta Z\left(1+Z\right)\right)\right)\right)=\left(1+\omega\right)\left(\frac{1-\omega}{1+\omega}-R\right), \qquad (27)
$$

which generalizes (23). Note that $Q^a(0)=0$, and$\frac{dQ^a(Z)}{dZ}>0$. To establish monotonicity, note that the solutions to $\frac{dQ^a(Z)}{dZ}=0$ are given by

$$
Z=\frac{1}{3\left(1+R\right)}\left(-R\pm\frac{1}{\omega\beta}\sqrt{-3\omega\beta\left(1+R\left(\left(2+\omega\right)+\left(1+\omega\left(1-\frac{\beta}{3}\right)\right)R\right)\right)}\right),
$$

which has no real solution in the relevant parameter range. Furthermore, $Q^a(1/2)-(1+\omega)\left(\frac{1-\omega}{1+\omega}-R\right)\geq 0$ with equality only if $\omega=R=0$. Finally, for $\underline{Z}\equiv\left(-\frac{1}{6}\sqrt[3]{44+12\sqrt{69}}+\frac{10}{3\sqrt[3]{44+12\sqrt{69}}}-\frac{1}{3}\right)\approx -0.569\,84$, $Q^a(\underline{Z})-(1+\omega)\left(\frac{1-\omega}{1+\omega}-R\right)<0\ \forall R>0$ and $\omega,\beta\in[0,1]$, and we can therefore conclude that

1. if $R<\frac{1-\omega}{1+\omega}$, then $Z\in(0,1/2]$;
2. if $R>\frac{1-\omega}{1+\omega}$, then $Z\in(-0.6,0)$; and
3. if $R=\frac{1-\omega}{1+\omega}$, then $Z=0$.

To see that $Z>\underline{Z}$, we note that

$$
\begin{aligned}
& Q^a(\underline{Z})-(1+\omega)\left(\frac{1-\omega}{1+\omega}-R\right)\\
= \;& \left(2\underline{Z}+1\right)R\left(\left(1-\omega\beta\right)+\omega\left(1-\beta\right)\right)+2\underline{Z}\left(1+\omega\beta\underline{Z}^2\right)-1\left(1-\omega\right)<0
\end{aligned}
$$

We note that $\frac{d\left(Q(Z;R)-(1+\omega)\left(\frac{1-\omega}{1+\omega}-R\right)\right)}{dR}=1+\omega+2Z\left(1+\omega\left(1+\beta Z\left(1+Z\right)\right)\right)$. This expression is positive if $Z>0$. Thus, if $Z>0$, $dZ/dR<0$. Furthermore, $\frac{dQ(Z;\omega)}{d\omega}>0$ while $\frac{d(1+\omega)\left(\frac{1-\omega}{1+\omega}-R\right)}{d\omega}$ falls, implying that the equilibrium value of Z falls in ω. Again, this follows from the fact that $\frac{dQ(Z;\omega)}{d\omega}=0$ has no real solutions in the relevant range of parameters. Finally, using the definition of X and $Z=\frac{\alpha_1}{2-\beta\alpha_1 w}w$, we have

$$
X=\alpha_0+\alpha_1\frac{2-\left(1+\beta\left(1-\alpha_0\right)\right)w}{2-\beta\alpha_1 w}
$$

so that

$$\alpha_0 = \frac{Xw - Z(2 - w(1+\beta))}{w(1+\beta Z)}.$$

From (26) we also have $\alpha_0 = \alpha_{00}X + \alpha_{01}$, implying

$$X = \frac{Z(2 - w(1+\beta)) + w(\alpha_{01} + \alpha_{01}\beta Z)}{w(1 - \alpha_{00} - \alpha_{00}\beta Z)}$$

and

$$\begin{aligned} b^a &= \frac{X}{1-Z}, \\ u^a &= 1 - e(b^a, b^a). \end{aligned}$$

The expression for b^a is, in general, a complicated function of Z. It simplifies considerably when $R = \frac{1-\omega}{1+\omega}$ (implying $Z = 0$). Then

$$b^a = \frac{R(1+\omega)^2}{\frac{2+\beta}{1+\beta} + \omega(1-\omega)}.$$

■

Proposition 5 *Assume $\lambda \le \beta$ (altruism, "al") and $0 \le R \le R_{\max}$. The political equilibrium with altruistic voting is characterized as follows:*

$$\begin{aligned} B^{alo}(u_t) &= \begin{cases} b^{alo} + \frac{\rho}{w}\left(u_t - \underline{u}^{alo}\right) & \text{if } u_t \ge \underline{u}^{alo} - \frac{w}{\rho} b^{alo} \\ 0 & \text{otherwise} \end{cases} \\ U^{alo}(b_t) &= \underline{u}^{alo} + \frac{w}{2 - \beta\rho}\left(b_t - b^{alo}\right), \end{aligned}$$

where

$$\rho = \frac{2Z}{1+\beta Z}$$

and Z is a constant (details in the proof) with the following properties: (i) $Z \in [-1/(2+\lambda), 1/2]$, (ii) $dZ/dR < 0$, (iii) $dZ/d\lambda < 0$, and (iv) $Z > 0$ iff $R > (\beta - \lambda)/(\beta + \lambda)$.
For $t > 0$, the equilibrium law of motion is

$$u_{t+1} = \underline{u}^{alo} + Z\left(u_t - \underline{u}^{alo}\right).$$

Given any u_0, the economy converges to a unique steady state such that $b = b^{alo} \le b^{ao}$ and $u = \underline{u}^{alo} \le \underline{u}^{ao}$ following an oscillating (monotone) path if $R > (<) \frac{\beta-\lambda}{\beta+\lambda}$. If $R = 0$, $Z = \frac{1}{2}\left(1 - \frac{\lambda}{\beta}\right) \ge 0$ and $b^{alo} = 0$. If $R = \frac{\beta-\lambda}{\beta+\lambda}$, convergence is immediate to a steady state with $\frac{1+\beta}{\beta(2+\beta)-\lambda}(\beta - \lambda) \ge 0$.

Proof. The weighted average felicity $F(u_t, b_t, b_{t+1})$ is

$$\begin{aligned} F(u_t, b_t, b_{t+1}) = {} & \beta(\mu(1-u_t)w + \mu u_t b_t w) \\ & + (\beta+\lambda)((1-\mu)b_t w + R b_t w) \\ & + \lambda\left(\mu e(b_t, b_{t+1}) w + \mu(1 - e(b_t, b_{t+1})) b_t w - \mu e(b_t, b_{t+1})^2\right) \\ & - (\lambda+\beta)(1+R)\tau(u_t, b_t, b_{t+1}). \end{aligned}$$

The usual guess, $b_t = \alpha_0 + \alpha_1 u_t$, together with the equation for optimal effort choice yields $b_{t+1} = X + Zb_t$, implying $b^{alo} = \frac{X}{1-Z}$ and $u^{alo} = 1 - e\left(\frac{X}{1-Z}, \frac{X}{1-Z}\right)$ where, as above, $X \equiv \alpha_0 + \alpha_1 \frac{2-(1+\beta(1-\alpha_0))w}{2-\beta\alpha_1 w}$ and $Z \equiv \frac{\alpha_1 w}{2-\beta\alpha_1 w}$, which implies $u_{t+1} = 1 - (1 + \beta(1-X) \ \ - (1+\beta Z) b_t) \frac{w}{2}$. The problem admits a recursive formulation of the following type:

$$\begin{aligned} W(u_t) &= \max_{b_t \in [0,\bar{b}]} \{F(u_t, b_t, b_{t+1}) + \lambda W(u_{t+1})\}, \\ \text{s.t. } b_{t+1} &= X + Zb_t, \\ u_{t+1} &= 1 - (1 + \beta(1-X) - (1+\beta Z) b_t) \frac{w}{2}. \end{aligned} \tag{28}$$

Given the guess, the first-order condition for maximizing the right-hand side of the Bellman equation is

$$\frac{\partial F}{\partial b_t} + \frac{\partial F}{\partial b_{t+t}} Z + \lambda W'(u_{t+1}) (1+\beta Z) \frac{w}{2} = 0,$$

where

$$\begin{aligned} \frac{\partial F}{\partial b_t} &= \beta\mu u_t w + (\beta+\lambda)((1-\mu) w + Rw) + \lambda(\mu(1-e_t) w) \\ &\quad - (\lambda+\beta)(1+R) \frac{\partial \tau}{\partial b_t} + \lambda\mu(w - b_t w - 2e_t) \frac{\partial e_t}{\partial b_t}, \\ \frac{\partial F}{\partial b_{t+t}} &= -(\lambda+\beta)(1+R) \frac{\partial \tau}{\partial b_{t+t}} + \lambda\mu(w - wb_t - 2e_t) \frac{\partial e_t}{\partial b_{t+1}}. \end{aligned}$$

We also know that

$$\begin{aligned} \frac{\partial e_t}{\partial b_t} &= -\frac{w}{2}, \quad \frac{\partial e_t}{\partial b_{t+1}} = -\beta\frac{w}{2}, \\ \frac{\partial \tau}{\partial b_t} &= \left(1 - \frac{\mu}{2}\left(1 - u_t + (1+\beta)\frac{w}{2} - (b_t + \beta(X + Zb_t))\frac{w}{2}\right)\right) w + \mu b_t \frac{w^2}{4} \\ \frac{\partial \tau}{\partial b_{t+1}} &= \frac{1}{4}\beta\mu w^2 b_t. \end{aligned}$$

Using the envelope condition and the fact that $\frac{\partial \tau(u_t, b_t, b_{t+1})}{\partial u_t} = \frac{1}{2}\mu b_t w$, we obtain

$$\begin{aligned} W'(u_t) &= \beta(-\mu w + \mu b_t w) - (\lambda+\beta)(1+R) \frac{\partial \tau(u_t, b_t, b_{t+1})}{\partial u_t} \\ &= \left(\frac{\beta-\lambda}{2} - \frac{\lambda+\beta}{2} R\right) \mu w b_t - \beta\mu w, \end{aligned}$$

implying that we can write the first-order condition as

$$\begin{aligned} &(\beta\mu u_t + (\beta+\lambda)(1-\mu+R) + \lambda\mu(1-e_t)) w \\ &- (\lambda+\beta)(1+R) \frac{\partial \tau}{\partial b_t} - \lambda\mu(w - b_t w - 2e_t) \frac{w}{2} \\ &- Z(\lambda+\beta)(1+R) \frac{\partial \tau}{\partial b_{t+1}} - Z\lambda\mu(w - wb_t - 2e_t) \beta\frac{w}{2} \\ &+ \lambda\left(\left(\frac{\beta-\lambda}{2} - \frac{\lambda+\beta}{2} R\right) \mu w b_t - \beta\mu w\right)(1+\beta Z)\frac{w}{2} \\ = \ & 0. \end{aligned}$$

Collecting terms and using $e_t = (1 + \beta (1 - X) \quad - (1 + \beta Z) b_t) \frac{w}{2}$ yields

$$0 = \frac{1}{2}\mu w^2 (1 + Z\beta) \left(\frac{1}{2}\lambda (\beta - \lambda) - \beta (1 + \lambda Z) - R (\beta + \lambda) \left(1 + \frac{1}{2}\lambda \right) \right) b_t + \frac{\mu w}{2} (\lambda + \beta) \left(\frac{\beta - \lambda}{\lambda + \beta} - R \right) u_t + C, \quad (29)$$

where

$$C = \left(1 + \frac{1}{2} (1 + \beta (1 - X)) w \right) \mu \frac{w}{2} (\beta + \lambda) R - (\beta - \lambda (1 - w)) \mu \frac{w}{2} - \frac{1}{2}\lambda \beta \mu w^2 (1 + Z (1 + \beta)) + \left(Z\lambda\beta + \frac{\beta + \lambda}{2} \right) \frac{\mu}{2} (1 + \beta (1 - X)) w^2.$$

The guess $b_t = \alpha_0 + \alpha_1 u_t$ and the first-order condition (29) can be used to solve, by equating coefficients, for α_0 and α_1. Next, using the definition $Z \equiv \frac{\alpha_1 w}{2 - \beta \alpha_1 w}$ implies that Z must satisfy

$$Q^{alo} (Z; R) \equiv Z \left(\beta (1 + \lambda Z) - \lambda \frac{\beta - \lambda}{2} + R \frac{\beta + \lambda}{2} (2 + \lambda) \right) = \frac{\lambda + \beta}{2} \left(\frac{\beta - \lambda}{\lambda + \beta} - R \right) \quad (30)$$

and that Z belongs to the unit interval. We then obtain equilibrium Z as the unique stable solution to the quadratic equation $Q^{alo} (Z; R) - \frac{\lambda + \beta}{2} \left(\frac{\beta - \lambda}{\lambda + \beta} - R \right) = 0$. This is given by

$$Z = \frac{ - \left(\beta - \lambda \frac{\beta - \lambda}{2} + R \frac{\beta + \lambda}{2} (2 + \lambda) \right)}{2\lambda\beta} + \frac{\sqrt{\left(\beta - \lambda \frac{\beta - \lambda}{2} + R \frac{\beta + \lambda}{2} (2 + \lambda) \right)^2 + 4\lambda\beta \frac{\lambda + \beta}{2} \left(\frac{\beta - \lambda}{\lambda + \beta} - R \right)}}{2\lambda\beta}.$$

The solution is always real, since the discriminant increases in R and equals $\frac{1}{4} (\lambda (\beta - \lambda) + 2\beta)^2 > 0$ when $R = 0$.[1] To see that the other root is smaller than -1, we note that when $R = 0$ it is given by $-\frac{1}{\lambda} < -1$ and that it decreases in R. Straightforward algebra then shows that for any $R \geq 0$, there is a solution to (30) in $\left[-\frac{1}{2+\lambda}, \frac{1}{2} \right]$, which decreases in R and λ, and that $Z < (\geq) 0$ if $R > (\leq) \frac{\beta - \lambda}{\lambda + \beta}$. Finally, from the definition of α_0 and α_1 we have $\alpha_0 = \frac{Xw - Z(2 - w(1+\beta))}{w(1+\beta Z)}$. Using this in the solution for the constant in (29), we obtain

$$(Xw - Z (2 - w (1 + \beta))) \frac{1}{2}\mu w \left(\frac{1}{2}\lambda (\beta - \lambda) - \beta (1 + \lambda Z) - R (\beta + \lambda) \left(1 + \frac{1}{2}\lambda \right) \right) = -C,$$

[1] The discriminant is

$$\left(\beta - \lambda \frac{\beta - \lambda}{2} + R \frac{\beta + \lambda}{2} (2 + \lambda) \right)^2 + 4\lambda\beta \frac{\lambda + \beta}{2} \left(\frac{\beta - \lambda}{\lambda + \beta} - R \right)$$

with a derivative

$$\frac{1}{2} (\beta + \lambda) \left((\lambda + \beta) \left(4 (1 + \lambda) + \lambda^2 \right) R + 4\beta (1 - \lambda) + \lambda^2 (2 - (\beta - \lambda)) \right).$$

from which we find

$$\begin{aligned}
X &= -\frac{-\left(\left(2+\left(1+\beta\right)w\right)\left(\beta+\lambda\right)\right)R}{w\left(\left(2+\lambda+\beta\right)\left(\beta+\lambda\right)R+\lambda^{2}+\beta\left(2+\beta\right)+2Z\lambda\beta\left(1+\beta\right)\right)} \\
&\quad +\frac{\left(Z+\frac{1}{2}\left(\frac{2-\lambda}{\lambda}+\frac{\lambda}{\beta}\right)+\frac{1}{2}\frac{2+\lambda}{\lambda\beta}\left(\beta+\lambda\right)R\right)2\left(2-w\left(1+\beta\right)\right)\lambda\beta Z}{w\left(\left(2+\lambda+\beta\right)\left(\beta+\lambda\right)R+\lambda^{2}+\beta\left(2+\beta\right)+2Z\lambda\beta\left(1+\beta\right)\right)} \\
&\quad -\frac{\left(\beta-\lambda\right)\left(2-w\left(1+\beta\right)\right)}{w\left(\left(2+\lambda+\beta\right)\left(\beta+\lambda\right)R+\lambda^{2}+\beta\left(2+\beta\right)+2Z\lambda\beta\left(1+\beta\right)\right)}.
\end{aligned}$$

To see the derivations of the claims on Z in more detail, we note

$$\begin{aligned}
Q^{alo}\left(0;R\right) &= 0, \\
\frac{dQ^{alo}\left(Z;R\right)}{dZ} &= R\frac{\beta+\lambda}{2}\left(2+\lambda\right)-\frac{1}{2}\lambda\left(\beta-\lambda\right)+\beta\left(1+2\lambda Z\right), \\
Q^{alo}\left(\frac{1}{2};R\right)-\frac{\lambda+\beta}{2}\left(\frac{\beta-\lambda}{\lambda+\beta}-R\right) &= \left(\beta+\lambda\right)\left(1+\frac{1}{4}\lambda\right)R+\frac{1}{2}\lambda\left(1+\frac{1}{2}\lambda\right) \\
&\geq 0,
\end{aligned}$$

with equality only if $\lambda=R=0$. Therefore, when $R=\frac{\beta-\lambda}{\lambda+\beta}$, it must be that $Z=0$ and that $b^{alo}=X=\frac{1+\beta}{\beta(2+\beta)-\lambda}\left(\beta-\lambda\right)$. In addition, $\frac{dQ^{alo}(Z;R)}{dZ}>0$ if $Z>0$ and $\frac{\beta-\lambda}{\lambda+\beta}>R$. Thus, $\frac{\beta-\lambda}{\lambda+\beta}>R$ implies $Z\in\left(0,\frac{1}{2}\right]$. Furthermore, since both

$$\frac{d\left(Q^{alo}\left(Z;R\right)-\frac{\lambda+\beta}{2}\left(\frac{\beta-\lambda}{\lambda+\beta}-R\right)\right)}{d\lambda}=\lambda+2\beta Z+\frac{1}{2}\left(1-\beta\right)+R\left(\frac{3}{2}+\lambda+\frac{1}{2}\beta\right)$$

and

$$\frac{d\left(Q^{alo}\left(Z;R\right)-\frac{\lambda+\beta}{2}\left(\frac{\beta-\lambda}{\lambda+\beta}-R\right)\right)}{dR}=\left(\beta+\lambda\right)\frac{1}{2}\left(1+Z\left(2+\lambda\right)\right)$$

are larger than zero when $Z>0$, the equilibrium value of Z is decreasing in λ and R. Next, consider the case when $\frac{\beta-\lambda}{\lambda+\beta}<R$. Clearly, now equilibrium Z is negative. Furthermore, for any $Z>-\frac{1}{2+\lambda}$,

$$\begin{aligned}
\frac{d\left(Q^{alo}\left(Z;R\right)-\frac{\lambda+\beta}{2}\left(\frac{\beta-\lambda}{\lambda+\beta}-R\right)\right)}{dZ} &= R\frac{\beta+\lambda}{2}\left(2+\lambda\right)-\frac{1}{2}\lambda\left(\beta-\lambda\right)+\beta\left(1+2\lambda Z\right) \\
&> \frac{\beta-\lambda}{\lambda+\beta}\frac{\beta+\lambda}{2}\left(2+\lambda\right)-\frac{1}{2}\lambda\left(\beta-\lambda\right)+\beta\left(1+2\lambda Z\right) \\
&= 2\beta\left(1+Z\lambda\right)-\lambda>0,
\end{aligned}$$

and since

$$\begin{aligned}
&\left(Q^{alo}\left(-\frac{1}{2+\lambda};R\right)-\frac{\lambda+\beta}{2}\left(\frac{\beta-\lambda}{\lambda+\beta}-R\right)\right) \\
&= -\left(\frac{1}{2+\lambda}\beta+\frac{1}{2}\left(\beta-\lambda\right)\right)\left(1-\frac{\lambda}{2+\lambda}\right)\leq 0,
\end{aligned}$$

equilibrium $Z\geq-\frac{1}{2+\lambda}$. Finally,

$$\begin{aligned}
\frac{d\left(Q^{alo}\left(Z;R\right)-\frac{\lambda+\beta}{2}\left(\frac{\beta-\lambda}{\lambda+\beta}-R\right)\right)}{d\lambda} &= \lambda+2\beta Z+\frac{1}{2}\left(1-\beta\right)+R\left(\frac{3}{2}+\lambda+\frac{1}{2}\beta\right) \\
&\geq \left(\lambda-2\beta\frac{1}{2+\lambda}+\frac{1}{2}\left(1-\beta\right)+\frac{\beta-\lambda}{\lambda+\beta}\left(\frac{3}{2}+\lambda+\frac{1}{2}\beta\right)\right)>0
\end{aligned}$$

and

$$\frac{d\left(Q^{alo}\left(Z;R\right)-\frac{\lambda+\beta}{2}\left(\frac{\beta-\lambda}{\lambda+\beta}-R\right)\right)}{dR}=\left(\beta+\lambda\right)\frac{1}{2}\left(1+Z\left(2+\lambda\right)\right),$$

implying that equilibrium Z is decreasing in R and in λ. ■

Proposition 6 *The optimal solution to the planner program (15) in the case $\lambda=\beta$ is*

$$b_t=b^p-\left(b_{t-1}-b^p\right),\forall t\geq 1,$$

and

$$b_0=\left(1+\frac{1-u_0}{\left(1-\beta\right)\frac{w}{2}}\right)b^p,$$

where

$$b^p\equiv\frac{R}{1+2R}.$$

Proof. We first state the value function, $V\left(b_t\right)$, which below will be shown to be the unique solution to (18) when $\lambda=\beta$,

$$V\left(b_t\right)=\frac{\mu w^2}{4}\left(A_0+A_1b_t+A_2b_t^2\right), \tag{31}$$

where

$$\begin{aligned}
A_2 &\equiv -\beta\left(1-\beta+2R\right),\\
A_1 &\equiv 2\beta\left(2R-\beta\right),\\
A_0 &\equiv \frac{\left(64R^2-9\beta^2\right)\beta^2}{16\left(1-\beta\right)\left(2R+1\right)}+\frac{4}{\left(1-\beta\right)\mu w^2}\tilde{Q},\\
\tilde{Q} &\equiv \frac{w^2}{4}\mu\left(1+\beta\right)\beta\left(3-\beta\right).
\end{aligned}$$

When $\lambda=\beta$, the current felicity Y in the recursive formulation (18) simplifies to

$$\begin{aligned}
Y\left(b_t,b_{t+1}\right) = &\left(\frac{\mu w^2}{4}\right)\cdot\{2\beta\left(2R\left(1+\beta\right)-\beta\right)b_t\\
&-\beta\left(1+2\left(1+\beta\right)R\right)b_t^2-2\beta^2\left(1+2R\right)b_tb_{t+1}\\
&+2\beta^3b_{t+1}-\beta^3b_{t+1}^2\}+\tilde{Q}.
\end{aligned}$$

Note that the right-hand side in the Bellman equation is concave in b_{t+1} since the coefficient on b_{t+1}^2, given by $-\beta^2\left(1+2R\right)$, is negative. The solution to $\frac{d}{db_{t+1}}\left\{Y\left(b_t,b_{t+1}\right)+\beta V\left(b_{t+1}\right)\right\}=0$ is given by

$$b_{t+1}=\frac{2R}{1+2R}-b_t<1,$$

so the condition $b_{t+1}\leq 1$ will never bind. The first-order condition $0\geq\frac{d}{db_{t+1}}\left\{Y\left(b_t,b_{t+1}\right)+\beta V\left(b_{t+1}\right)\right\}$ is therefore sufficient for optimality and its solution is given by

$$b_{t+1}=f\left(b_t\right)=\max\left\{0,b^*-\left(b_t-b^*\right)\right\},$$

where $b^* = R/(1+2R)$.[2] It is now straightforward to verify that $V(b) = Y(b, f(b)) + \beta V(f(b))$. This proves that V is a fixed-point of the functional mapping $\Gamma(v) = \max_{b' \in [0,1]} \{Y(b, b') + \lambda v(b')\}$. Since Γ is a contraction mapping, f must be the unique optimal policy. Consider now the first-period problem (17). Inserting the expression for Y_0 and V and simplifying yields

$$\begin{aligned} & \max_{b_0 \in [0,1]} \{Y_0(u_0, b_0) + V(b_0)\} \\ = \quad & \mu w \beta R \left(1 - u_0 + \frac{w}{2}(1-\beta)\right) b_0 - \frac{\mu w^2}{4} \beta (1-\beta)(2R+1) b_0^2 + \hat{Q}, \end{aligned}$$

where $\hat{Q}$ is a constant. This problem is convex, since the coefficient on b_0^2 is negative. The first-order condition $\frac{d}{db_0} \{Y_0(u_0, b_0) + V(b_0)\} \leq 0$ yields

$$b_0 = \min \left\{ \frac{R}{(2R+1)} \left(1 + \frac{(1-u_0)}{\frac{w}{2}(1-\beta)}\right), 1 \right\},$$

which is monotonically decreasing in u_0. ■

[2] Note that for all $t \geq 0$, $b_{t+1} = \frac{2R}{1+2R} - b_t \leq \frac{2R}{1+2R}$. Thus, $b_{t+2} = \frac{2R}{1+2R} - b_{t+1} \geq \frac{2R}{1+2R} - \frac{2R}{1+2R} = 0$. In other words, the constraint $b_t \geq 0$ cannot be strictly binding for $t \geq 2$.