Appendix B (not for publication)

7.1 Statement and Proof of Lemma 7

Lemma 7 *The program (10) subject to (4) and (5) is a contraction mapping. Hence, a solution exists and is unique.*

Proof. Consider the intra-temporal FOC, (13), that is derived in the text. The condition solves

$$g = \Theta(\tau), \tag{64}$$

with $\Theta'(\cdot) < 0$. We can rewrite the government budget constraint as

$$b' - Rb = \Lambda(\tau) \equiv \Theta(\tau) - \tau w H(\tau),$$

where $\Lambda : [0, \bar{\tau}] \to [-\bar{\tau} w H(\bar{\tau}), \Theta(0)]$ is monotonic. Therefore, $\tau = \Lambda^{-1}(b' - Rb)$. Then, (10) can be rewritten as

$$V_O^{comm}(b) = \max_{b' \in [\underline{b}, \bar{b}]} \left\{ \hat{v}\left(b' - Rb\right) + \beta\lambda V_O^{comm}\left(b'\right) \right\}, \tag{65}$$

where

$$\hat{v}\left(b' - Rb\right) \equiv (1+\lambda)\, u\left(\Theta\left(\Lambda^{-1}\left(b' - Rb\right)\right)\right) + \lambda\phi\left(A\left(\Lambda^{-1}\left(b' - Rb\right)\right)\right).$$

Since the function $\hat{v}$ is bounded and continuous, and $\beta\lambda < 1$, Theorem 4.6 in Stokey and Lucas (1989) establishes that (65) has a unique fixed point. ■

7.2 Statement and Proof of Lemma 8

Lemma 8 *Assume that* $\lambda > 0$, *and*

$$(1+\lambda)\left(\left(\left(\Lambda^{-1}\right)'\right)^2\left(u''\left(\Theta'\right)^2 + u'\left(\Theta''\right)\right) + u'\Theta'\left(\Lambda^{-1}\right)''\right)$$
$$+\lambda\left(\left(\left(\Lambda^{-1}\right)'\right)^2\left(\phi''\left(A'\right)^2 + \phi' A''\right) + \phi' A'\left(\Lambda^{-1}\right)''\right) < 0, \tag{66}$$

where Λ *is defined in the proof of Lemma 7. Then, the unique MPPE of Lemma 7 is a DMPPE.*

Proof. The proof is an application of Theorem 2.1 in Santos (1991).[35] The theorem states that the policy functions are differentiable if (i) the return function $\hat{v}$ is strictly concave and (ii) optimal paths are strictly interior. Consider the formulation of the problem used in the proof of Lemma 7. Standard differentiation shows that the function $\hat{v}$ is strictly concave if and only if assumption (66) holds.

We must show that the optimal paths are interior; i.e., that for any $b \in \left(\underline{b}, \bar{b}\right)$,

$$B(b) \in \left(\underline{b}, \bar{b}\right).$$

[35] Santos, Manuel "Smoothness of the Policy Function in Discrete Time Economic Models," *Econometrica*, 59 (1991), 1365-1382.

Since setting $B(b) = \bar{b}$ would imply zero public expenditure in the next period, then $\lambda > 0$ ensures that $B(b) < \bar{b}$. It remains to prove that $B(b) > \underline{b}$ for any $b \in (\underline{b}, \bar{b})$. Suppose instead that there exists a $\hat{b} \in (\underline{b}, \bar{b})$ such that $B\left(\hat{b}\right) = \underline{b}$. The Euler equation must then be (recall that $\hat{v}'' < 0$);

$$\hat{v}'\left(\underline{b} - R\hat{b}\right) \leq \beta\lambda R\hat{v}'\left(B\left(\underline{b}\right) - R\underline{b}\right). \tag{67}$$

By concavity of $\hat{v}$, $\hat{b} > \underline{b}$ implies $\hat{v}'\left(\underline{b} - R\underline{b}\right) < \hat{v}'\left(\underline{b} - R\hat{b}\right)$. Equation (67) then implies

$$\hat{v}'\left(\underline{b} - R\underline{b}\right) < \beta\lambda R\hat{v}'\left(B\left(\underline{b}\right) - R\underline{b}\right) \tag{68}$$

Thus, if the agent is constrained for some $\hat{b} > \underline{b}$, she must be constrained for $b = \underline{b}$. Hence, $B\left(\underline{b}\right) = \underline{b}$. However, $\beta\lambda R \leq 1$ implies $\hat{v}'\left(\underline{b} - R\underline{b}\right) \geq \beta\lambda R\hat{v}'\left(B\left(\underline{b}\right) - R\underline{b}\right)$, which contradicts (68). This concludes the proof. ■

7.3 Statement and Proof of Proposition 3

For convenience, we restate the Proposition 3 already contained in the text.

Proposition 8 *[RESTATEMENT OF Proposition 3] Let $\left\langle \bar{B}(b), \bar{G}(b), \bar{T}(b)\right\rangle$ denote equilibrium policies when $\omega = 1$. Assume that $\bar{B}(b), \bar{G}(b), \bar{T}(b)$ are continuously differentiable. If*

$$\left| \frac{u''(g)}{\beta\lambda R u''(g')} - \bar{G}'(b')\left(1 - \frac{\phi'(A(\tau))\, u''(g)\, wH(\tau)\,(1 - e(\tau))\,/u'(g)}{\phi''(A(\tau))\, A'(\tau) + \phi'(A(\tau))\, e'(\tau)\,/\,(1 - e(\tau))}\right)\right| > 1,$$

where $g = \bar{G}(b)$, $g' = \bar{G}(b')$, $\tau = \bar{T}(b)$ and $b' = \bar{B}(b)$. Then, for ω close to unity, there exists a unique DMPPE.

Proof. The strategy of the proof is based on Judd (2004). Let $\left\langle \bar{B}(b), \bar{G}(b), \bar{T}(b)\right\rangle$, denote the equilibrium policies when $\omega = 1$. Time subscripts will denote partial derivatives. We rewrite the equilibrium conditions (15), (19) and (18) in the following form:

$$\frac{u'\left(\bar{G}(b)\right)}{u'\left(\bar{G}\left(\hat{B}\left(\bar{G}(b), b\right)\right)\right)} = \beta\lambda R - \beta\lambda G'\left(\hat{B}(\bar{G}(b), b)\right)\frac{1-\omega}{\lambda(1+\lambda\omega)}, \tag{69}$$

$$\hat{B}\left(\bar{G}(b), b\right) = \bar{G}(b) + Rb - \hat{T}\left(\bar{G}(b)\right)\cdot w \cdot H\left(\hat{T}\left(\bar{G}(b)\right)\right), \tag{70}$$

$$\phi'\left(A\left(\hat{T}\left(\bar{G}(b)\right)\right)\right) = \frac{1+\omega\lambda}{1-\omega(1-\lambda)}\left(1 - e\left(\hat{T}\left(\bar{G}(b)\right)\right)\right)u'\left(\bar{G}(b)\right). \tag{71}$$

Let $\varepsilon \equiv \frac{\beta(1-\omega)}{\lambda(1+\lambda\omega)}$ where $\lim_{\omega\to 1}\varepsilon = 0$. We prove that in the neighborhood of $\varepsilon = 0$ there exists a unique policy function $G(b,\varepsilon)$ that solves the GEE, (69). Note that $G(b,\varepsilon)$ involves some slight abuse of notation. We plug-in the candidate equilibrium function $G(b,\varepsilon)$ into (69), obtaining

$$\Pi(\varepsilon, G(b,\varepsilon)) \equiv \frac{u'(G(b,\varepsilon))}{u'\left(G\left(\hat{B}(G(b,\varepsilon), b), \varepsilon\right)\right)} - \beta\lambda R + \varepsilon G_1\left(\hat{B}(G(b,\varepsilon), b), \varepsilon\right) = 0, \tag{72}$$

where we define

$$\hat{B}\left(G\left(b,\varepsilon\right),b\right)=G\left(b,\varepsilon\right)+Rb-\hat{T}\left(G\left(b,\varepsilon\right)\right)H\left(\hat{T}\left(G\left(b,\varepsilon\right)\right)\right). \tag{73}$$

Next, we differentiate (72) with respect to ε, and evaluate the resulting expression at $\varepsilon=0$ (recalling that $G\left(b,0\right)=\bar{G}\left(b\right)$ and $G_1\left(b,0\right)=\bar{G}'\left(b\right)$).

$$\begin{aligned}
&\frac{u''\left(\bar{G}\left(b\right)\right)G_2\left(b,0\right)}{u'\left(\bar{G}\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\right)}\\
&-\frac{u'\left(\bar{G}\left(b\right)\right)u''\left(\bar{G}\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\right)}{\left(u'\left(\bar{G}\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\right)\right)^2}\left(\begin{array}{c}\bar{G}'\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\hat{B}_1\left(\bar{G}\left(b\right),b\right)G_2\left(b,0\right)\\ +G_2\left(\hat{B}\left(\bar{G}\left(b\right),b\right),0\right)\end{array}\right)\\
&+\bar{G}'\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)=0
\end{aligned}$$

After rearranging terms and using the fact that $u'\left(\bar{G}\left(b\right)\right)=\beta\lambda Ru'\left(\bar{G}\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\right)$ as implied by the GEE when $\varepsilon=0$, we obtain:

$$\begin{aligned}
&-\frac{\beta\lambda Ru''\left(\bar{G}\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\right)}{u'\left(\bar{G}\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\right)}G_2\left(\hat{B}\left(\bar{G}\left(b\right),b\right),0\right)\\
&+\left(\begin{array}{c}-\frac{\beta\lambda Ru''\left(\bar{G}\left(\hat{B}\left(\bar{G}(b),b\right)\right)\right)}{u'\left(\bar{G}\left(\hat{B}\left(\bar{G}(b),b\right)\right)\right)}\bar{G}'\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\hat{B}_1\left(\bar{G}\left(b\right),b\right)\\ +\frac{u''\left(\bar{G}(b)\right)}{u'\left(\bar{G}\left(\hat{B}\left(\bar{G}(b),b\right)\right)\right)}\end{array}\right)G_2\left(b,0\right)\\
&+\bar{G}'\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)=0
\end{aligned} \tag{74}$$

Therefore, (74) implies that:

$$G_2\left(b,0\right)=J\left(b\right)\cdot G_2\left(\hat{B}\left(\bar{G}\left(b\right),b\right),0\right)+Z\left(b\right), \tag{75}$$

where

$$\begin{aligned}
J\left(b\right)&=\left(\frac{u''\left(\bar{G}\left(b\right)\right)}{\beta\lambda Ru''\left(\bar{G}\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\right)}-\bar{G}'\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\hat{B}_1\left(\bar{G}\left(b\right),b\right)\right)^{-1}\\
Z\left(b\right)&=J\left(b\right)\cdot\bar{G}'\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\left(-\frac{u'\left(\bar{G}\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\right)}{\beta\lambda Ru''\left(\bar{G}\left(\hat{B}\left(\bar{G}\left(b\right),b\right)\right)\right)}\right)
\end{aligned}$$

Note that (75) has an iterative nature. Define the mapping:

$$\left(\Upsilon\ G_2\left(b,0\right)\right)\left(b\right)\equiv J\left(b\right)\cdot G_2\left(\hat{B}\left(\bar{G}\left(b\right),b\right),0\right)+Z\left(b\right).$$

If $\left|J\left(b\right)\right|<1$, then Υ is a contraction mapping. We now show $\left|J\left(b\right)\right|<1$ if and only if assumption (20) holds. Differentiating equation (71), we solve

$$\hat{T}'\left(\bar{G}\left(b\right)\right)=\frac{\frac{1+\omega\lambda}{1-\omega(1-\lambda)}\left(1-e\left(\bar{T}\left(b\right)\right)\right)u''\left(\bar{G}\left(b\right)\right)}{\phi''\left(A\left(\bar{T}\left(b\right)\right)\right)A'\left(\bar{T}\left(b\right)\right)+\frac{1+\omega\lambda}{1-\omega(1-\lambda)}e'\left(\bar{T}\left(b\right)\right)u'\left(\bar{G}\left(b\right)\right)}. \tag{76}$$

Differentiating equation (70), together with (76), leads to

$$\begin{aligned}\hat{B}_1(G(b),b) &= 1-\hat{T}'\left(\bar{G}(b)\right)wH\left(\bar{T}(b)\right)\left(1-e\left(\bar{T}(b)\right)\right)\\ &= 1-\frac{\frac{1+\omega\lambda}{1-\omega(1-\lambda)}\left(1-e\left(\bar{T}(b)\right)\right)u''\left(\bar{G}(b)\right)wH\left(\bar{T}(b)\right)\left(1-e\left(\bar{T}(b)\right)\right)}{\phi''\left(A\left(\bar{T}(b)\right)\right)A'\left(\bar{T}(b)\right)+\frac{1+\omega\lambda}{1-\omega(1-\lambda)}e'\left(\bar{T}(b)\right)u'\left(\bar{G}(b)\right)}.\end{aligned}$$

Hence

$$|J(b)|=\left|\left(\frac{u''\left(\bar{G}(b)\right)}{\beta\lambda Ru''\left(\bar{G}\left(\hat{B}\left(\bar{G}(b),b\right)\right)\right)}-\bar{G}'\left(\hat{B}\left(\bar{G}(b),b\right)\right)\cdot\hat{B}_1\left(\bar{G}(b),b\right)\right)^{-1}\right|<1,$$

by assumption (20) and the intra-temporal FOC (18) when $\omega=1$. This establishes that Υ is a contraction mapping. Therefore, in the neighbuorhood of $\omega=1$, there exists a unique derivative $G_2(b,0)$.

Finally, we must show that the existence of a unique derivative $G_2(b,0)$ establishes the existence of a unique equilibrium policy function, $G(b,\varepsilon)$, that satisfies the GEE. Differentiating the functional equation (72) with respect to ε and evaluating the result at $\varepsilon=0$ lead to the linear operator equation

$$\Pi_1(0,G(b,0))+\Pi_2(0,G(b,0))\,G_2(b,0)=0.$$

The existence and the uniqueness of $G_2(b,0)$ imply that $\Pi_2(0,G(b,0))$ is invertible at neighborhood of $\varepsilon=0$. Therefore, we can apply implicit function theorem (Judd, 2004, pp. 10) to show that there are neighborhoods ε_0 of $\varepsilon=0$ and for all $\varepsilon\in\varepsilon_0$, there is a unique $G(b,\varepsilon)$. ■

7.4 Analysis of Example II (Section 4.2)

In this section we provide a formal analysis of Example II in section 4.2. The next Proposition provides an analytical characterization of the MPPE.

Definition 3 *Let* $\psi\equiv\omega/(1-\omega(1-\lambda))\in(0,1/\lambda)$ *and let* $b_0^*\equiv\bar{b}\left(1-\frac{\theta(1-\bar{\tau})}{(1-\omega)\bar{\tau}(1+\beta)}\right)$ *(note that in the paper this is referred to as* b^**).*

Before proceeding, we rewrite the program (16) as follows.

$$\arg\max\left\{\hat{v}(\tau,g)-(1-\psi\lambda)\,\theta\log g+\beta\lambda V_O\left(b'\right)\right\}, \tag{77}$$

where $\hat{v}(\tau,g)\equiv(1+\lambda)\,\theta\log g+(1+\beta)\,\lambda\log A(\tau)$, and V_O satisfies the following functional equation;

$$V_O\left(b'\right)=\hat{v}\left(T\left(b'\right),G\left(b'\right)\right)+\beta\lambda V_O\left(B\left(b'\right)\right).$$

Proposition 9 *Assume that* $\theta<\frac{1+\beta}{(1+\lambda)\beta}\left(1-\lambda\left(\beta+\frac{1+\psi}{1+\lambda}\right)\right)$ *and* $R\in\left[1+\frac{1+\psi}{(1+\lambda)\beta},\frac{1+\beta+\theta(1+\psi)}{((1+\lambda)\theta+(1+\beta)\lambda)\beta}\right]$*. Then, there exists* $\underline{b}<b_0^*$ *such that for* $b\in[\underline{b},\bar{b}]$*, a MPPE is characterized by the following policy functions:*

$$\tau=T(b)\equiv\begin{cases}\bar{\tau}-\frac{R(1+\beta)}{w(1+\beta+\theta(1+\psi))}\left(b_0^*-b\right) & \text{if } b\in[\underline{b},b_0^*)\\ \bar{\tau} & \text{otherwise}\end{cases}, \tag{78}$$

$$g = G(b) \equiv \begin{cases} g_0^* + \frac{\theta(1+\psi)R}{1+\beta+\theta(1+\psi)}\left(b_0^* - b\right) & \text{if } b \in [\underline{b}, b_0^*) \\ b_n^* + \bar{\tau} w - Rb & \text{if } b \in \left[b_n^*, b_{n+1}^*\right) \end{cases}, \tag{79}$$

$$b' = B(b) \equiv \begin{cases} b_0^* \equiv \bar{b}\left(1 - \frac{\theta(1+\psi)(1-\bar{\tau})}{\bar{\tau}(1+\beta)}\right) & \text{if } b \in [\underline{b}, b_1^*) \\ b_n^* & \text{if } b \in \left[b_n^*, b_{n+1}^*\right] \end{cases}, \tag{80}$$

where $\bar{b} \equiv \bar{\tau} w / (R-1)$, $g_0^* \equiv w\theta(1+\psi)(1-\bar{\tau})/(1+\beta) > 0$, *and the sequence* $\{b_n^*\}_{n=0,1,2,..,\infty}$ *is the unique solution to the difference equation*

$$\left(b_n^* - b_{n+1}^* + \bar{\tau} w\right)^{1+\psi}\left(b_n^* - Rb_n^* + \bar{\tau} w\right)^{(1+\lambda)\beta} = \left(b_{n+1}^* - Rb_{n+1}^* + \bar{\tau} w\right)^{1+\psi+(1+\lambda)\beta}, \tag{81}$$

given b_0^*. *The sequence* $\{b_n^*\}_{n=0,1,2,..,\infty}$ *is monotonically increasing in* n *and* $\lim_{n\to\infty} b_n^* = \bar{b}$.

The equilibrium is shown in Figure A1.

FIGURE A1 HERE

Proof Strategy. The proof strategy is analogous to Krusell and Smith (2003). We have structured the proof in 10 lemmas. Lemmas 9 establishes that the set of admissible interest rates, $R \in \left[1 + \frac{1+\psi}{(1+\lambda)\beta}, \frac{1+\beta+\theta(1+\psi)}{((1+\lambda)\theta+(1+\beta)\lambda)\beta}\right]$ is nonempty. Lemma 10 establishes that the sequence $\{b_n^*\}_{n=0}^{\infty}$ implicitly defined by (81) describing the set of steady states converges to $\bar{b}$ along an increasing path. Lemma 11-18 jointly establish that there exist no profitable one-period deviation from the equilibrium. In other words, they show that if future policy outcomes follow the equilibrium functions (78), (79) and (80), the current political choice is also characterized by (78), (79) and (80) (namely, we have a fixed point). Specifically, Lemma 11 checks the intra-temporal optimality condition and Lemma 12 provides the objective function of the current government's intertemporal choices. Lemma 13 proves then that there exists $\underline{b} < b_0^*$ such that for $b \in [\underline{b}, b_0^*]$, b_0^* and b_n^* are locally optimal for $b' \in [\underline{b}, b_0^*)$ and $b' \in \left[b_n^*, b_{n+1}^*\right)$, respectively. Next, Lemma 16 shows that b_0^* is globally optimal for $b \in [\underline{b}, b_0^*]$. We then move to $b \in [b_0^*, b_1^*]$. The local optimality of b_n^* for $b' \in \left[b_n^*, b_{n+1}^*\right)$ is obtained by Lemma 14. The global optimality of $b' = b_0^*$ is ensured by Lemma 15 and 18, showing that $b' = b_0^*$ is better than any $b' > b_0^*$ and any $b' < b_0^*$, respectively. Using Lemma 17, the same logic applies for $b \in \left[b_n^*, b_{n+1}^*\right]$ with $n \geq 1$.

Lemma 9 $\left[1 + \frac{1+\psi}{(1+\lambda)\beta}, \frac{1+\beta+\theta(1+\psi)}{((1+\lambda)\theta+(1+\beta)\lambda)\beta}\right]$ *is nonempty if* $\theta < \frac{1+\beta}{(1+\lambda)\beta}\left(1 - \lambda\left(\beta + \frac{1+\psi}{1+\lambda}\right)\right)$.

Proof. Some algebra establishes that if $\theta < \frac{1+\beta}{(1+\lambda)\beta}\left(1 - \lambda\left(\beta + \frac{1+\psi}{1+\lambda}\right)\right)$,

$$\frac{1+\beta+\theta(1+\psi)}{((1+\lambda)\theta+(1+\beta)\lambda)\beta} > 1 + \frac{1+\psi}{(1+\lambda)\beta}$$

always holds. ■

Lemma 10 *The sequence* $\{b_n^*\}_{n=0}^{\infty}$ *converges to* $\bar{b}$ *along an increasing path.*

Proof. (81) gives an implicit difference equation of b_n^*. Rearranging (81) and using the fact that $\bar{\tau}w = (R-1)\bar{b}$, we obtain

$$y_n = (x_n)^{-\frac{(1+\lambda)\beta}{1+\psi}}, \tag{82}$$

where $y_n \equiv \frac{(R-1)\bar{b}+b_n^*-Rb_{n+1}^*}{(R-1)\left(\bar{b}-b_{n+1}^*\right)}$ and $x_n \equiv \frac{\bar{b}-b_n^*}{\bar{b}-b_{n+1}^*}$. Linearizing (82) around $b_n^* = b_{n+1}^*$ yields

$$y_n - 1 = -\frac{(1+\lambda)\beta}{1+\psi}(x_n - 1),$$

or equivalently

$$\frac{b_n^* - b_{n+1}^*}{(R-1)\left(\bar{b}-b_{n+1}^*\right)} = \frac{(1+\lambda)\beta}{1+\psi}\frac{\left(b_n^* - b_{n+1}^*\right)}{\bar{b}-b_n^*}.$$

This establishes

$$b_{n+1}^* = \bar{b} - \frac{1+\psi}{(1+\lambda)\beta(R-1)}\left(\bar{b}-b_n^*\right). \tag{83}$$

It is immediate that if $\frac{1+\psi}{(1+\lambda)\beta(R-1)} < 1$ (or equivalently $R > 1 + \frac{1+\psi}{(1+\lambda)\beta}$), b_n^* is converging to the maximum debt level $\bar{b}$ along an increasing path. ∎

Lemma 11 *Suppose that* $B(b)$ *follows (80). Then, the optimal intra-temporal solution is such that* $\tau = 1 - \frac{1+\beta}{\theta(1+\psi)w}g \leq \bar{\tau}$ *if* $b \leq b_0^*$ *and* $\tau = \bar{\tau}$ *otherwise.*

Proof. The first-order condition linking τ and g establishes

$$\tau = \begin{cases} 1 - \frac{1+\beta}{\theta(1+\psi)w}g & \text{if } g \geq \frac{(1-\bar{\tau})\theta(1+\psi)w}{1+\beta} \\ \bar{\tau} & \text{otherwise} \end{cases}.$$

Given $B(b)$, the above equality leads to $G(b)$. Replacing g with $G(b)$, we obtain an equivalence between $g \geq \frac{(1-\bar{\tau})\theta(1+\psi)w}{1+\beta}$ and $b \leq b_0^*$. ∎

Lemma 12 *Suppose that future policy outcomes follow (78), (79) and (80). Then, the current government's objective function is*

$$V\left(b';b\right) = \begin{cases} \begin{array}{c} (1+\beta+\theta(1+\psi))\log(b'+w-Rb) \\ +((1+\lambda)\theta+(1+\beta)\lambda)\beta\log(b_0^*+w-Rb') \\ +\theta\zeta\beta\lambda\log(\bar{\tau}w-(R-1)b_0^*) \end{array} & \text{if } b' \in [\underline{b}, b_0^*] \\ \\ \begin{array}{c} (1+\beta+\theta(1+\psi))\log(b'+w-Rb) \\ +(1+\lambda)\theta\beta\log(b_n^*-Rb'+\bar{\tau}w) \\ +\theta\zeta\beta\lambda\log(\bar{\tau}w-(R-1)b_n^*) \end{array} & \text{if } b' \in \left[b_n^*, b_{n+1}^*\right] \end{cases} \tag{84}$$

for $b \in [\underline{b}, b_0^*]$ *and*

$$V\left(b';b\right) = \begin{cases} \begin{array}{c} (1+\psi)\theta\log(b'+w-Rb) \\ +((1+\lambda)\theta+(1+\beta)\lambda)\beta\log(b_0^*+w-Rb') \\ +\theta\zeta\beta\lambda\log(\bar{\tau}w-(R-1)b_0^*) \end{array} & \text{if } b' \in [\underline{b}, b_0^*] \\ \\ \begin{array}{c} (1+\psi)\theta\log(b'+w-Rb) \\ +(1+\lambda)\theta\beta\log(b_n^*+\bar{\tau}w-Rb') \\ +\theta\zeta\beta\lambda\log(\bar{\tau}w-(R-1)b_n^*) \end{array} & \text{if } b' \in \left[b_n^*, b_{n+1}^*\right] \end{cases} \tag{85}$$

for $b \geq b_0^*$*, where* $\zeta \equiv \frac{(1+\lambda)\beta}{1-\beta\lambda}$.

Proof. (78), (79) and (80) establish that for $i \geq 1$, $\tau_{t+i} = \bar{\tau}$ and $b_{t+1+i} = b_n^*$ if $b_{t+1} \in \left[b_n^*, b_{n+1}^*\right]$. If $b_{t+1} \in [\underline{b}, b_0^*]$, we have $\tau_{t+1} = \bar{\tau} - \frac{R(1+\beta)}{w(1+\beta+\theta(1+\psi))}\left(b_0^* - b_{t+1}\right)$, $\tau_{t+i} = \bar{\tau}$ and $b_{t+i} = b_0^*$ for $i \geq 2$. Denoting b' as the current government's choice of public debt and ignoring constant terms, we thus obtain

$$V_O\left(b'\right) = \begin{cases} \begin{array}{c}\left(\left(1+\lambda\right)\theta + \left(1+\beta\right)\lambda\right)\log\left(b_0^* - Rb' + w\right) \\ +\frac{(1+\lambda)\theta}{1-\beta\lambda}\beta\lambda\log\left(\bar{\tau}w - \left(R-1\right)b_0^*\right)\end{array} & \text{if } b' \in [\underline{b}, b_0^*] \\ \\ \begin{array}{c}\left(1+\lambda\right)\theta\log\left(b_n^* - Rb' + \bar{\tau}w\right) \\ +\frac{(1+\lambda)\theta}{1-\beta\lambda}\beta\lambda\log\left(\bar{\tau}w - \left(R-1\right)b_n^*\right)\end{array} & \text{if } b' \in \left[b_n^*, b_{n+1}^*\right] \end{cases}$$

Substituting the above equation into (77) and using Lemma 11 lead to (84) and (85). ■

Lemma 13 *Suppose that future policy outcomes follow (78), (79) and (80). Then, there exists* $\underline{b} < b_0^*$ *such that for* $b \in [\underline{b}, b_0^*]$*, any choice* $b' \in [\underline{b}, b_0^*)$ *can be improved by* $b' = b_0^*$ *and any choice* $b' \in \left(b_n^*, b_{n+1}^*\right)$ *can be improved by* $b' = b_n^*$.

Proof. For $b \in [\underline{b}, b_0^*]$, since $\frac{\partial V}{\partial b'}$ is increasing in b and decreasing in b', it is sufficient to show that

$$\left.\frac{\partial V}{\partial b'}\right|_{b=\underline{b}, b'=b_0^*} = \frac{1+\beta+\theta\left(1+\psi\right)}{b_0^* + w - R\underline{b}} - \frac{\left(\left(1+\lambda\right)\theta + \left(1+\beta\right)\lambda\right)\beta R}{b_0^* + w - Rb_0^*} \geq 0.$$

Since $R \leq \frac{1+\beta+\theta(1+\psi)}{((1+\lambda)\theta+(1+\beta)\lambda)\beta}$, there exists $\underline{b} < b_0^*$ such that the above inequality always holds. Hence, any choice $b' \in [\underline{b}, b_0^*]$ can be improved by $b' = b_0^*$.

For $b' \in \left[b_n^*, b_{n+1}^*\right]$, it is sufficient to prove that

$$\left.\frac{\partial V}{\partial b'}\right|_{b=b_0^*, b'=b_n^*} = \frac{1+\beta+\theta\left(1+\psi\right)}{b_n^* + w - Rb_0^*} - \frac{\left(1+\lambda\right)\theta\beta R}{b_n^* + \bar{\tau}w - Rb_n^*} \leq 0.$$

Since $\left.\frac{\partial V}{\partial b'}\right|_{b=b_0^*, b'=b_n^*}$ is decreasing in b_n^*, we only need to show

$$\frac{1+\beta+\theta\left(1+\psi\right)}{b_0^* + w - Rb_0^*} \leq \frac{\theta\left(1+\lambda\right)\beta R}{b_0^* + \bar{\tau}w - Rb_0^*}.$$

The above inequality implies that

$$\begin{aligned}\left(1+\beta+\theta\left(1+\psi\right)\right)\bar{\tau} - \theta\left(1+\lambda\right)\beta R &\leq \left(1+\beta+\theta\left(1+\psi\right) - \theta\left(1+\lambda\right)\beta R\right)\left(R-1\right)\frac{b_0^*}{w} \\ &= \left(1+\beta+\theta\left(1+\psi\right) - \theta\left(1+\lambda\right)\beta R\right)\left(\frac{\left(1+\beta+\theta\left(1+\psi\right)\right)\bar{\tau} - \theta\left(1+\psi\right)}{1+\beta}\right)\end{aligned}$$

First note that when $\bar{\tau} = 1$, LHS is equal to RHS. For the inequality to hold for $\bar{\tau} \in [0, 1)$, we need to show that the slope of LHS $1+\beta+\theta\left(1+\psi\right)$ is greater than the slope of RHS $\frac{(1+\beta+\theta(1+\psi)-\theta(1+\lambda)\beta R)(1+\beta+\theta(1+\psi))}{1+\beta}$. This is ensured by the fact that $R > 1 + \frac{1+\psi}{(1+\lambda)\beta}$. Hence, any $b' \in \left(b_n^*, b_{n+1}^*\right)$ with $n \geq 0$ can be improved by b_n^*. ■

Lemma 14 *Suppose that future policy outcomes follow (78), (79) and (80). Then, for* $b \in [b_0^*, b_n^*]$*, any choice* $b' \in \left(b_n^*, b_{n+1}^*\right)$ *can be improved by* $b' = b_n^*$.

Proof. Since the tax rate is constrained when $b \geq b_0^*$, the government's objective function follows (85). For $b' \geq b_0^*$, differentiating V with respect to b' yields

$$\frac{\partial V}{\partial b'} = \frac{(1+\psi)\,\theta}{b' - Rb + \bar{\tau}w} - \frac{\theta\,(1+\lambda)\,\beta R}{b_n^* - Rb' + \bar{\tau}w}.$$

It is sufficient to prove that $\frac{\partial V}{\partial b'} \leq 0$ at $b' = b_n^*$ for $b \in [b_0^*, b_n^*]$.

$$\left.\frac{\partial V}{\partial b'}\right|_{b'=b_n^*} = \frac{(1+\psi)\,\theta}{b_n^* - Rb + \bar{\tau}w} - \frac{\theta\,(1+\lambda)\,\beta R}{b_n^* - Rb_n^* + \bar{\tau}w} \leq 0$$

We show that

$$\begin{aligned}
\frac{1+\psi}{b_n^* - Rb + \bar{\tau}w} &\leq \frac{(1+\lambda)\,\beta R}{b_n^* - Rb_n^* + \bar{\tau}w} \\
(1+\psi)\,(b_n^* + \bar{\tau}w) - (1+\psi)\,Rb_n^* &\leq (1+\lambda)\,\beta R\,(b_n^* + \bar{\tau}w) - (1+\lambda)\,\beta R^2 b \\
(1+\psi - (1+\lambda)\,\beta R)\,(b_n^* + \bar{\tau}w) &\leq (1+\psi - (1+\lambda)\,\beta R)\,Rb_n^* - (1+\lambda)\,\beta R^2\,(b - b_n^*) \\
((1+\lambda)\,\beta R - 1 - \psi)\,(b_n^* - Rb_n^* + \bar{\tau}w) &\geq (1+\lambda)\,\beta R^2\,(b - b_n^*)
\end{aligned}$$

This is always true if $b \leq b_n^*$. ■

Lemma 15 *Suppose that future policy outcomes follow (78), (79) and (80). Then, for any* $b \in \left[b_n^*, b_{n+1}^*\right]$, $b' = b_{n+s}$ *for any* $s > 0$ *is dominated by* $b' = b_n^*$.

Proof. We show that for any $b \in \left[b_n^*, b_{n+1}^*\right]$,

$$\begin{aligned}
&\theta\,(1+\psi)\log\left(b_n^* - Rb + \bar{\tau}w\right) + \theta\,(1+\lambda)\,\beta\log\left(b_n^* - Rb_n^* + \bar{\tau}w\right) \\
> \;&\theta\,(1+\psi)\log\left(b_{n+s}^* - Rb + \bar{\tau}w\right) + \theta\,(1+\lambda)\,\beta\log\left(b_{n+s} - Rb_{n+s}^* + \bar{\tau}w\right). \qquad (86)
\end{aligned}$$

Rearrange

$$\begin{aligned}
&\theta\,(1+\psi)\left(\log\left(b_n^* - Rb + \bar{\tau}w\right) - \log\left(b_{n+s}^* - Rb + \bar{\tau}w\right)\right) \\
> \;&\theta\,(1+\lambda)\,\beta\left(\log\left(b_{n+s}^* - Rb_{n+s}^* + \bar{\tau}w\right) - \log\left(b_n^* - Rb_n^* + \bar{\tau}w\right)\right)
\end{aligned}$$

The LHS and RHS of this expression can be written as

$$\sum_{i=0}^{i=s-1} \theta\,(1+\psi)\left(\log\left(b_{n+i}^* - Rb + \bar{\tau}w\right) - \log\left(b_{n+i+1}^* - Rb + \bar{\tau}w\right)\right) \qquad (87)$$

$$\sum_{i=0}^{i=s-1} \theta\,(1+\lambda)\,\beta\left(\log\left(b_{n+i+1}^* - Rb_{n+i+1}^* + \bar{\tau}w\right) - \log\left(b_{n+i}^* - Rb_{n+i}^* + \bar{\tau}w\right)\right). \qquad (88)$$

According to the difference equation

$$\left(b_n^* - Rb_{n+1}^* + \bar{\tau}w\right)^{1+\psi}\left(b_n^* - Rb_n^* + \bar{\tau}w\right)^{(1+\lambda)\beta} = \left(b_{n+1}^* - Rb_{n+1}^* + \bar{\tau}w\right)^{1+\psi+(1+\lambda)\beta}.$$

(88) is equal to

$$\sum_{i=0}^{i=s-1} \theta (1+\psi) \left(\log \left(b_{n+i}^* - Rb_{n+i+1}^* + \bar{\tau} w\right) - \log \left(b_{n+i}^* - Rb_{n+i}^* + \bar{\tau} w\right)\right)$$

Due to the concavity of the log function and the increasing b_n^*,

$$\begin{aligned} & \log \left(b_{n+i}^* - Rb + \bar{\tau} w\right) - \log \left(b_{n+i+1}^* - Rb + \bar{\tau} w\right) \\ > \; & \log \left(b_{n+i}^* - Rb_{n+i+1}^* + \bar{\tau} w\right) - \log \left(b_{n+i+1}^* - Rb_{n+i+1}^* + \bar{\tau} w\right) \\ > \; & \log \left(b_{n+i}^* - Rb_{n+i+1}^* + \bar{\tau} w\right) - \log \left(b_{n+i}^* - Rb_{n+i}^* + \bar{\tau} w\right) \end{aligned}$$

for any $b \in \left[b_n^*, b_{n+1}^*\right]$. This establishes that the LHS of (86) is indeed larger than the RHS of (86). ■

Lemma 16 *Suppose that future policy outcomes follow (78), (79) and (80). Then, for* $b \in [\underline{b}, b_0^*]$*,* $b' = b_s^*$ *for any* $s > 0$ *is dominated by* $b' = b_0^*$*.*

Proof. We need to show that for any $b \in \left[b_n^*, b_{n+1}^*\right]$,

$$\begin{aligned} & (1+\beta+\theta(1+\psi)) \log (b_0^* - Rb + w) + \theta (1+\lambda) \beta \log (b_0^* - Rb_0^* + \bar{\tau} w) \\ > \; & (1+\beta+\theta(1+\psi)) \log (b_s^* - Rb + w) + \theta (1+\lambda) \beta \log (b_s^* - Rb_s^* + \bar{\tau} w) . \end{aligned}$$

The rest of the proof simply follows the same procedure as in Lemma 15. ■

Lemma 17 *Suppose* $z_H > z_L$ *and* $b_H > b_L$*. Then,*

$$\begin{aligned} & \theta (1+\psi) \log (z_H - Rb_L + \bar{\tau} w) + \theta (1+\lambda) \beta \log (B(z_H) - Rz_H + \bar{\tau} w) \\ > \; & \theta (1+\psi) \log (z_L - Rb_L + \bar{\tau} w) + \theta (1+\lambda) \beta \log (B(z_L) - Rz_L + \bar{\tau} w) \end{aligned}$$

implies that

$$\begin{aligned} & \theta (1+\psi) \log (z_H - Rb_H + \bar{\tau} w) + \theta (1+\lambda) \beta \log (B(z_H) - Rz_H + \bar{\tau} w) \\ > \; & \theta (1+\psi) \log (z_L - Rb_H + \bar{\tau} w) + \theta (1+\lambda) \beta \log (B(z_L) - Rz_L + \bar{\tau} w) . \end{aligned}$$

Proof. We need to show that

$$\begin{aligned} \log (z_H - Rb_L + \bar{\tau} w) &> \log (z_L - Rb_L + \bar{\tau} w) \Rightarrow \\ \log (z_H - Rb_H + \bar{\tau} w) &> \log (z_L - Rb_H + \bar{\tau} w) \end{aligned}$$

Define

$$F(b) \equiv \log (z_H - Rb + \bar{\tau} w) - \log (z_L - Rb + \bar{\tau} w)$$

It is straightforward that F is increasing in b since $z_L < z_H$. Hence, if $F(b_L) > 0$, $F(b_H)$ must be positive. ■

Lemma 18 *Suppose $z_H > z_L$ and $b_H > b_L$. Then,*

$$\begin{aligned} & \left(1+\beta+\theta\left(1+\psi\right)\right)\log\left(z_H - Rb_L + \bar{\tau}w\right) + \theta\left(1+\lambda\right)\beta\log\left(B\left(z_H\right) - Rz_H + \bar{\tau}w\right) \\ > & \left(1+\beta+\theta\left(1+\psi\right)\right)\log\left(z_L - Rb_L + \bar{\tau}w\right) + \theta\left(1+\lambda\right)\beta\log\left(B\left(z_L\right) - Rz_L + \bar{\tau}w\right) \end{aligned}$$

implies that

$$\begin{aligned} & \theta\left(1+\psi\right)\log\left(z_H - Rb_H + \bar{\tau}w\right) + \theta\left(1+\lambda\right)\beta\log\left(B\left(z_H\right) - Rz_H + \bar{\tau}w\right) \\ > & \theta\left(1+\psi\right)\log\left(z_L - Rb_H + \bar{\tau}w\right) + \theta\left(1+\lambda\right)\beta\log\left(B\left(z_L\right) - Rz_L + \bar{\tau}w\right). \end{aligned}$$

Proof. The proof follows the same procedure as in Lemma 17. ■

Now we can prove Proposition 9.

Proof. We start with $b \in [\underline{b}, b_0^*]$, Lemma 13 and 16 imply that the optimal $b' = b_0^*$. Then we move to $b \in [b_0^*, b_1^*]$, Lemma 14 and 15 establish that $b' = b_0^*$ is better than any $b' \geq b_0^*$. Moreover, Lemma 18 establishes that any choice $b' < b_0^*$ cannot be optimal. So the optimal solution $b' = b_0^*$. The proof is completed by following the same procedure for any $n \geq 1$ (using Lemma 17). ■

7.5 Measurement of Public Goods

Our empirical measure of public good provision in the U.S. (from footnote 21) encompasses the following expenditure items: defense, highways, libraries, hospitals, health, employment security administration, veterans' services, air transportation, water transport and terminals, parking facilities, transit subsidies, police protection, fire protection, correction, protective inspection and regulation, sewerage, natural resources, parks and recreation, housing and community development, solid waste management, financial administration, judicial and legal, general public buildings, other government administration, and other general expenditures, not elsewhere classified.

Consumption is total personal consumption expenditures. The data source is the Economic Report of the President, tables B1, B20, and B86.

7.6 Statement and Proof of Proposition 10

Claim 1 *The DMPPE is defined by the program:*

$$\left\langle G\left(s_{-1},b\right), T\left(s_{-1},b\right)\right\rangle = \arg\max_{\tau,g} \quad \tilde{\nu}\left(\tau,g,s;s_{-1}\right) + \left(\tilde{\delta}-1\right)\lambda u\left(A\left(\tau\right)-s,g\right) + \tilde{\delta}\beta\lambda V_O\left(s,b'\right),$$

where $\tilde{\nu}\left(\tau,g,s;s_{-1}\right) = u\left(Rs_{-1},g\right) + \lambda u\left(A\left(\tau\right)-s,g\right)$, $\tilde{\delta} = 1 + \left(1-\omega\right)/\left(\lambda\omega\right)$, $g' = G\left(s,b'\right)$, $s = \tilde{S}\left(\tau,g,g'\right)$, *and* $b' = Rb + g - \tau w H\left(\tau\right)$. V_O *is a fixed point of the following functional equation:*

$$V_O\left(s_{-1},b\right) = \tilde{\nu}\left(T\left(s_{-1},b\right), G\left(s_{-1},b\right), S\left(s_{-1},b\right); s_{-1}\right) + \beta\lambda V_O\left(S\left(s_{-1},b\right), B\left(s_{-1},b\right)\right),$$

where S satisfies the equation:

$$S\left(s_{-1},b\right) = \tilde{S}\left(T\left(s_{-1},b\right), G\left(s_{-1},b\right), G\left(S\left(s_{-1},b\right), B\left(s_{-1},b\right)\right)\right).$$

Proof. We write the political objective function in a sequential formulation.

$$\frac{\lambda}{1-\omega+\omega\lambda}U = \frac{\omega\lambda}{1-\omega+\omega\lambda}u\left(Rs_{-1},g_0\right) + \lambda U_Y\left(\boldsymbol{s},\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{g}\right)$$

$$= \frac{\omega\lambda}{1-\omega+\omega\lambda}u\left(Rs_{-1},g_0\right) + \lambda\sum_{t=0}^{\infty}\left(\lambda\beta\right)^t\left(u\left(A\left(\tau_t\right)-s_t,g_t\right)+\beta u\left(Rs_t,g_{t+1}\right)\right)$$

$$\begin{aligned} = \; & \frac{\omega\lambda}{1-\omega+\omega\lambda}u\left(Rs_{-1},g_0\right) + \lambda u\left(A\left(\tau_0\right)-s_0,g_0\right) \\ & + \sum_{t=1}^{\infty}\left(\lambda\beta\right)^t\left(u\left(Rs_{t-1},g_t\right)+\lambda u\left(A\left(\tau_t\right)-s_t,g_t\right)\right). \end{aligned}$$

Multiplying both sides by $\frac{1-\omega+\omega\lambda}{\lambda\omega}$ yields

$$= \tilde{\nu}\left(\tau_0,g_0,s_0;s_{-1}\right) + \left(\tilde{\delta}-1\right)\lambda u\left(A\left(\tau_0\right)-s_0,g_0\right) + \tilde{\delta}\sum_{t=1}^{\infty}\left(\lambda\beta\right)^t\cdot\tilde{\nu}\left(\tau_t,g_t,s_t;s_{t-1}\right)$$

$$= \tilde{\nu}\left(\tau_0,g_0,s_0;s_{-1}\right) + \left(\tilde{\delta}-1\right)\lambda u\left(A\left(\tau_0\right)-s_0,g_0\right) + \tilde{\delta}\lambda\beta\cdot V_O\left(s_0,b_1\right),$$

where

$$\begin{aligned} V_O\left(s_0,b_1\right) &= \sum_{t=1}^{\infty}\left(\lambda\beta\right)^t\tilde{\nu}\left(\tau_t,g_t,s_t;s_{t-1}\right) \\ &= \sum_{t=1}^{\infty}\left(\lambda\beta\right)^t\tilde{\nu}\left(T\left(s_{t-1},b_t\right),G\left(s_{t-1},b_t\right),s_t;s_{t-1}\right), \end{aligned}$$

$s_t = S\left(s_{t-1},b_t\right)$ and $b_{t+1} = B\left(s_{t-1},b_t\right)$. The second equality follows from the fact that future variables must follow the equilibrium policy rules. Note in particular that in equilibrium, saving satisfies

$$S\left(s_{-1},b\right) = \tilde{S}\left(T\left(s_{-1},b\right),G\left(s_{-1},b\right),G\left(S\left(s_{-1},b\right),B\left(s_{-1},b\right)\right)\right).$$

Since $\beta\lambda < 1$, V_O can be expressed recursively as

$$V_O\left(s_{-1},b\right) = \tilde{\nu}\left(T\left(s_{-1},b\right),G\left(s_{-1},b\right),S\left(s_{-1},b\right);s_{-1}\right) + \beta\lambda V_O\left(S\left(s_{-1},b\right),B\left(s_{-1},b\right)\right).$$

This concludes the proof of the preliminaries. ■

Proposition 10 *Let $u = u(c,g)$, where $u_c > 0$, $u_g > 0$, and u is a quasi-concave function. Then, a DMPPE is characterized by a system of two functional equations:*

1. *A trade-off between private and public good consumption*

$$\begin{aligned} \tilde{\delta}\lambda u_1\left(c_Y,g\right)A'\left(\tau\right) = \; & \left(u_2\left(Rs_{-1},g\right)+\tilde{\delta}\lambda u_2\left(c_Y,g\right)\right) \\ & \cdot\left(1-e\left(\tau\right)\right)A'\left(\tau\right) \\ & +\beta\lambda\left(\tilde{\delta}-1\right)\frac{u_2\left(Rs,g'\right)G_1\left(s,b'\right)}{1-\tilde{S}_3\left(\tau,g,g'\right)G_1\left(s,b'\right)} \\ & \cdot\left(\tilde{S}_2\left(\tau,g,g'\right)\left(1-e\left(\tau\right)\right)A'\left(\tau\right)-\tilde{S}_1\left(\tau,g,g'\right)\right). \end{aligned} \quad (89)$$

where subscripts denote partial derivatives, and the following equilibrium conditions hold

$$\begin{aligned}
c_Y &= A(\tau) - s,\ c'_Y = A(\tau') - s', \\
c'_O &= Rs,\ c''_O = Rs', \\
g &= G(s_{-1}, b),\ g' = G(s, b'),\ g'' = G(s', b'') \\
\tau &= T(s_{-1}, b),\ \tau' = T(s, b'), \\
s &= \tilde{S}(\tau, g, g'),\ s' = \tilde{S}(\tau', g', g''), \\
b' &= g + Rb - \tau w H(\tau) \equiv B(s_{-1}, b),\ b'' = B(s, b').
\end{aligned}$$

2. *A Generalized Euler Equation (GEE) for public good consumption:*

$$\begin{aligned}
&\frac{u_2(Rs_{-1}, g) + \tilde{\delta}\lambda u_2(c_Y, g)}{u_2(Rs, g')} \\
&+\beta\lambda\left(\tilde{\delta} - 1\right)\left(\frac{G_2(s, b') + G_1(s, b')\tilde{S}_2(\tau, g, g')}{1 - \tilde{S}_3(\tau, g, g') G_1(s, b')}\right) \\
= \; & R\beta\lambda\left(\begin{array}{c} 1 + \tilde{\delta}\lambda\frac{u_2(c'_Y, g')}{u_2(c'_O, g')} \\ +\beta\lambda\left(\tilde{\delta} - 1\right)\frac{u_2(c''_O, g'')}{u_2(c'_O, g')} G_1(s', b'')\frac{\tilde{S}_2(\tau', g', g'') + \tilde{S}_3(\tau', g', g'') G_2(s', b'')}{1 - \tilde{S}_3(\tau', g', g'') G_1(s', b'')} \end{array}\right).
\end{aligned} \tag{90}$$

where c_Y, c'_Y, c'_O, c''_O, g, g', g'', τ, τ', s, s', b' and b'' are equilibrium values defined as above.

Proof. We start the proof from an analysis of the effect of τ and g on private savings. Taking the total differential of the saving function, $s = \tilde{S}(\tau, g, g') = \tilde{S}(\tau, g, G(s, b'))$, with respect to τ and g yields, respectively,

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}\tau} &= \tilde{S}_1(\tau, g, g') + \tilde{S}_3(\tau, g, g')\left(G_1(s, b')\frac{\mathrm{d}s}{\mathrm{d}\tau} - G_2(s, b')(1 - e(\tau)) w H(\tau)\right) \Rightarrow \\
\frac{\mathrm{d}s}{\mathrm{d}\tau} &= \frac{\tilde{S}_1(\tau, g, g') - \tilde{S}_3(\tau, g, g') G_2(s, b')(1 - e(\tau)) w H(\tau)}{1 - \tilde{S}_3(\tau, g, g') G_1(s, b')},
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}g} &= \tilde{S}_2(\tau, g, g') + \tilde{S}_3(\tau, g, g')\left(G_1(s, b')\frac{\mathrm{d}s}{\mathrm{d}g} + G_2(s, b')\right) \Rightarrow \\
\frac{\mathrm{d}s}{\mathrm{d}g} &= \frac{\tilde{S}_2(\tau, g, g') + \tilde{S}_3(\tau, g, g') G_2(s, b')}{1 - \tilde{S}_3(\tau, g, g') G_1(s, b')},
\end{aligned}$$

where we note that

$$\frac{\frac{\mathrm{d}s}{\mathrm{d}\tau}}{(1 - e(\tau)) w H(\tau)} + \frac{\mathrm{d}s}{\mathrm{d}g} = \frac{\frac{\tilde{S}_1(\tau, g, g')}{(1 - e(\tau)) w H(\tau)} + \tilde{S}_2(\tau, g, g')}{1 - \tilde{S}_3(\tau, g, g') G_1(s, b')}. \tag{91}$$

Now consider the problem defined in Claim 1. The FOC w.r.t. τ is:

$$0 = \tilde{\delta}\lambda u_1(A(\tau) - s, g)\left(A'(\tau) - \frac{\mathrm{d}s}{\mathrm{d}\tau}\right) + \tilde{\delta}\beta\lambda V_{O1}(s, b')\frac{\mathrm{d}s}{\mathrm{d}\tau} - \tilde{\delta}\beta\lambda V_{O2}(s, b')(1 - e(\tau)) w H(\tau). \tag{92}$$

The FOC w.r.t. g is

$$0=\tilde{\delta}\lambda u_1\left(A\left(\tau\right)-s,g\right)\left(-\frac{\mathrm{d}s}{\mathrm{d}g}\right)+\tilde{\delta}\lambda u_2\left(A\left(\tau\right)-s,g\right)+u_2\left(Rs_{-1},g\right)+\tilde{\delta}\beta\lambda V_{O1}\left(s,b'\right)\frac{\mathrm{d}s}{\mathrm{d}g}+\tilde{\delta}\beta\lambda V_{O2}\left(s,b'\right). \tag{93}$$

We first derive (89), and then derive (90).

Derivation of (89). We claim (proof below):

$$V_{O1}\left(s_{-1},b\right)=u_1\left(Rs_{-1},g\right)R+\left(1-\frac{1}{\tilde{\delta}}\right)u_2\left(Rs_{-1},g\right)G_1\left(s_{-1},b\right). \tag{94}$$

Next, we combine (92) and (93) to substitute out $V_{O2}\left(s,b'\right)$:

$$\begin{aligned}&\frac{\tilde{\delta}\lambda u_1\left(A\left(\tau\right)-s,g\right)\left(-A'\left(\tau\right)+\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)-\tilde{\delta}\beta\lambda V_{O1}\left(s,b'\right)\frac{\mathrm{d}s}{\mathrm{d}\tau}}{\left(1-e\left(\tau\right)\right)wH\left(\tau\right)}\\ =\;&\tilde{\delta}\lambda u_1\left(A\left(\tau\right)-s,g\right)\left(-\frac{\mathrm{d}s}{\mathrm{d}g}\right)+\tilde{\delta}\lambda u_2\left(A\left(\tau\right)-s,g\right)+u_2\left(Rs_{-1},g\right)\\ &+\tilde{\delta}\beta\lambda V_{O1}\left(s,b'\right)\frac{\mathrm{d}s}{\mathrm{d}g}\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\tilde{\delta}\lambda u_1\left(c_Y,g\right)A'\left(\tau\right)\;=\;&\left(u_2\left(Rs_{-1},g\right)+\tilde{\delta}\lambda u_2\left(c_Y,g\right)\right)\\ &\cdot\left(1-e\left(\tau\right)\right)A'\left(\tau\right)\\ &+\beta\lambda\left(\tilde{\delta}-1\right)\frac{u_2\left(Rs,g'\right)G_1\left(s,b'\right)}{1-\tilde{S}_3\left(\tau,g,g'\right)G_1\left(s,b'\right)}\\ &\cdot\left(\tilde{S}_2\left(\tau,g,g'\right)\left(1-e\left(\tau\right)\right)A'\left(\tau\right)-\tilde{S}_1\left(\tau,g,g'\right)\right).\end{aligned}$$

where the first step uses (94) and the second step follows from equation (91) and from the household Euler equation, $u_1\left(A\left(\tau\right)-s,g\right)=\beta Ru_1\left(Rs,g'\right)$. The last expression is the "trade-off between private and public good consumption", (89).

Derivation of (90). We claim (proof below):

$$V_{O2}\left(s_{-1},b\right)=\left(1-\frac{1}{\tilde{\delta}}\right)u_2\left(Rs_{-1},g\right)G_2\left(s_{-1},b\right)+\beta\lambda RV_{O2}\left(s,b'\right). \tag{95}$$

Next, we use (94) to substitute out $V_{O1}\left(s,b'\right)$ in use (93), and use the household Euler equation, $u_1\left(A\left(\tau\right)-s,g\right)=\beta Ru_1\left(Rs,g'\right)$, to simplify terms. This yields:

$$\begin{aligned}0\;=\;&\tilde{\delta}\lambda u_2\left(A\left(\tau\right)-s,g\right)+u_2\left(Rs_{-1},g\right)\\ &+\beta\lambda\left(\tilde{\delta}-1\right)u_2\left(Rs,g'\right)G_1\left(s,b'\right)\frac{\mathrm{d}s}{\mathrm{d}g}+\tilde{\delta}\beta\lambda V_{O2}\left(s,b'\right).\end{aligned}$$

(95) implies that

$$
\begin{aligned}
& \frac{\tilde{\delta}\lambda u_2(c_Y,g)+u_2(Rs_{-1},g)}{u_2(Rs,g')} \\
& +\beta\lambda\left(\tilde{\delta}-1\right)\left(\frac{G_2(s,b')+G_1(s,b')\tilde{S}_2(\tau,g,g')}{1-\tilde{S}_3(\tau,g,g')G_1(s,b')}\right) \\
= \; & R\beta\lambda\left(\begin{array}{c} 1+\tilde{\delta}\lambda\frac{u_2\left(c'_Y,g'\right)}{u_2\left(c'_O,g'\right)} \\ +\beta\lambda\left(\tilde{\delta}-1\right)\frac{u_2\left(c''_O,g''\right)}{u_2\left(c'_O,g'\right)}G_1(s',b'')\frac{\tilde{S}_2(\tau',g',g'')+\tilde{S}_3(\tau',g',g'')G_2(s',b'')}{1-\tilde{S}_3(\tau',g',g'')G_1(s',b'')}\end{array}\right).
\end{aligned}
$$

This expression is the GEE for public good consumption, (89). ■

7.6.1 Derivation of equations (94) and (95)

Claim 2 *The partial derivatives* $V_{O1}(s_{-1},b)$ *and* $V_{O2}(s_{-1},b)$ *can be expressed as:*

$$V_{O1}(s_{-1},b)=u_1(Rs_{-1},g)R+\left(1-\frac{1}{\tilde{\delta}}\right)u_2(Rs_{-1},g)G_1(s_{-1},b).$$

$$V_{O2}(s_{-1},b)=\left(1-\frac{1}{\tilde{\delta}}\right)u_2(Rs_{-1},g)G_2(s_{-1},b)+\beta\lambda RV_{O2}\left(s,b'\right).$$

Proof. Differentiating $V_O(s_{-1},b)$ w.r.t. s_{-1} yields

$$
\begin{aligned}
V_{O1}(s_{-1},b) \; = \; & \lambda u_1(A(\tau)-s,g)\left(A'(\tau)-\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)T_1(s_{-1},b) \\
& +\lambda\left(u_1(A(\tau)-s,g)\left(-\frac{\mathrm{d}s}{\mathrm{d}g}\right)+u_2(A(\tau)-s,g)\right)G_1(s_{-1},b) \\
& +u_1(Rs_{-1},g)R+u_2(Rs_{-1},g)G_1(s_{-1},b) \\
& +\beta\lambda V_{O1}\left(s,b'\right)\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}T_1(s_{-1},b)+\frac{\mathrm{d}s}{\mathrm{d}g}G_1(s_{-1},b)\right) \\
& +\beta\lambda V_{O2}\left(s,b'\right)\left(G_1(s_{-1},b)-(1-e(\tau))wH(\tau)T_1(s_{-1},b)\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \left(\begin{array}{c}\lambda u_1(A(\tau)-s,g)\left(A'(\tau)-\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)+\beta\lambda V_{O1}(s,b')\frac{\mathrm{d}s}{\mathrm{d}\tau} \\ -\beta\lambda V_{O2}(s,b')(1-e(\tau))wH(\tau)\end{array}\right)T_1(s_{-1},b) \\
& +u_1(Rs_{-1},g)R+u_2(Rs_{-1},g)G_1(s_{-1},b) \\
& +\left(\begin{array}{c}-\lambda u_1(A(\tau)-s,g)\frac{\mathrm{d}s}{\mathrm{d}g}+\lambda u_2(A(\tau)-s,g) \\ +\beta\lambda V_{O1}(s,b')\frac{\mathrm{d}s}{\mathrm{d}g}+\beta\lambda V_{O2}(s,b')\end{array}\right)G_1(s_{-1},b).
\end{aligned}
$$

$$=u_1(Rs_{-1},g)R+\left(1-\frac{1}{\tilde{\delta}}\right)u_2(Rs_{-1},g)G_1(s_{-1},b),$$

which is equation (94).
Similarly, differentiating $V_O(s_{-1}, b)$ w.r.t. b yields

$$\begin{aligned} V_{O2}(s_{-1}, b) &= \lambda u_1 (A(\tau) - s, g) \left(A'(\tau) - \frac{\mathrm{d}s}{\mathrm{d}\tau} \right) T_2(s_{-1}, b) \\ &+ \lambda \left(u_1 (A(\tau) - s, g) \left(-\frac{\mathrm{d}s}{\mathrm{d}g} \right) + u_2 (A(\tau) - s, g) \right) G_2(s_{-1}, b) \\ &+ u_2 (Rs_{-1}, g) G_2(s_{-1}, b) \\ &+ \beta\lambda V_{O1}(s, b') \left(\frac{\mathrm{d}s}{\mathrm{d}\tau} T_2(s_{-1}, b) + \frac{\mathrm{d}s}{\mathrm{d}g} G_2(s_{-1}, b) \right) \\ &+ \beta\lambda V_{O2}(s, b') \left(-(1 - e(\tau)) wH(\tau) T_2(s_{-1}, b) + G_2(s_{-1}, b) + R \right) \end{aligned}$$

$$\begin{aligned} &= \begin{pmatrix} \lambda u_1 (A(\tau) - s, g) \left(A'(\tau) - \frac{\mathrm{d}s}{\mathrm{d}\tau} \right) + \\ \beta\lambda V_{O1}(s, b') \frac{\mathrm{d}s}{\mathrm{d}\tau} - \beta\lambda V_{O2}(s, b') (1 - e(\tau)) wH(\tau) \end{pmatrix} T_2(s_{-1}, b) \\ &+ \beta\lambda R V_{O2}(s, b') \\ &+ \begin{pmatrix} \lambda u_1 (A(\tau) - s, g) \left(-\frac{\mathrm{d}s}{\mathrm{d}g} \right) + \lambda u_2 (A(\tau) - s, g) \\ + \beta\lambda V_{O1}(s, b') \frac{\mathrm{d}s}{\mathrm{d}g} + \beta\lambda V_{O2}(s, b') + u_2 (Rs_{-1}, g) \end{pmatrix} G_2(s_{-1}, b) \\ &= \left(1 - \frac{1}{\bar{\bar{\delta}}} \right) u_2 (Rs_{-1}, g) G_2(s_{-1}, b) + \beta\lambda R V_{O2}(s, b'), \end{aligned}$$

which is equation (95). ■

7.7 Characterization of the Intrahousehold Allocation of Consumption (section 6)

Denote x the bequests that the young receive from the old. Then, the utility of the young can be written as

$$U_Y(\boldsymbol{x}, \boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{g}) = \log(c_Y) + \theta \log(g) + \beta \left(\log(c'_O) + \theta \log(g') + \lambda U_Y(\boldsymbol{x'}, \boldsymbol{b'}, \boldsymbol{\tau'}, \boldsymbol{g'}) \right).$$

They maximize U_Y subject to

$$\begin{aligned} c_Y &= x + A(\tau) - s, \\ c'_O &= Rs - x'. \end{aligned}$$

Differentiating $U_Y(\boldsymbol{x}, \boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{g})$ w.r.t. x, plus the standard Envelope argument, yields

$$\frac{\partial U_Y(\boldsymbol{x}, \boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{g})}{\partial x} = \frac{1 + \beta}{x + A(\tau) - x'/R}. \tag{96}$$

Next, consider the optimal bequest problem. The utility of the old is

$$U_O(s_{-1}, \boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{g}) = \log(c_O) + \theta \log(g) + \lambda U_Y(\boldsymbol{x}, \boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{g}),$$

where

$$c_O = Rs_{-1} - x.$$

Given (96) and the fact that $c_Y = (x + A(\tau) - x'/R) / (1+\beta)$, the FOC of the above problem implies

$$\frac{c_Y}{c_O} = \lambda. \tag{97}$$

Therefore, $c_Y = \lambda(1+\lambda)^{-1} c$, and $c_O = (1+\lambda)^{-1} c$, where c denotes total household consumption.

7.8 Two-Sided Altruism

In this section, we show how the sequential formulation of the political objective function is modified if the young care about the old (two-sided altruism). We start with the recursive formulation of the utility of each group:

$$U_Y(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{g}) = \phi(A(\tau)) + u(g) + \lambda_Y(\tilde{u}(c_O(A(\tau_{-1}))) + u(g)) + \beta\left(u(g') + \lambda U_Y(\boldsymbol{b}', \boldsymbol{\tau}', \boldsymbol{g}')\right) \tag{98}$$

$$U_O(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{g}) = \tilde{u}(c_O(A(\tau_{-1}))) + u(g) + \lambda U_Y(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{g}). \tag{99}$$

Note that the utility of the old, (99), is the same as that in the text. Instead, the utility of the young features the additional term $\lambda_Y(\tilde{u}(c_O(A(\tau_{-1}))) + u(g))$. λ_Y is the altruistic factor of the young towards the old.

Then, the political objective function (ignoring irrelevant constant terms involving $\tilde{u}(c_O(A(\tau_{-1})))$) can be written as

$$\begin{aligned}
\frac{\lambda}{1-\omega+\omega\lambda} U_0 &= \frac{\lambda}{1-\omega+\omega\lambda}((1-\omega)U_{Y,0} + \omega U_{O,0}) \\
&= \frac{\lambda}{1-\omega+\lambda\omega}(\omega u(g_0) + (1-\omega+\lambda\omega)U_{Y,0}) \\
&= \lambda\phi(A(\tau_0)) + \left(1 + \lambda_Y + \frac{\omega}{1-\omega+\lambda\omega}\right)\lambda u(g_0) \\
&\quad + \sum_{t=1}^{\infty}(\lambda\beta)^t(\lambda\phi(A(\tau_t)) + (1+\lambda(1+\lambda_Y))u(g_t)).
\end{aligned}$$

Multiplying both sides by $\frac{1+\lambda(1+\lambda^y)}{\lambda\left(1+\lambda^y+\frac{\omega}{1-\omega+\lambda\omega}\right)}$ yields

$$U_0 = \hat{\delta}\lambda\phi(A(\tau_0)) + (1+\lambda)u(g_0) + \hat{\delta}\sum_{t=1}^{\infty}(\lambda\beta)^t\hat{v}(g_t, \tau_t),$$

where

$$\hat{v}(\tau_t, g_t) = \lambda\phi(A(\tau_t)) + (1+\lambda(1+\lambda_Y))u(g_t),$$

$$\begin{aligned}
\hat{\delta} &\equiv \frac{1+\lambda(1+\lambda_Y)}{\lambda\left(1+\lambda_Y+\frac{\omega}{1-\omega+\lambda\omega}\right)} \\
&= 1 + \frac{1/\lambda - \frac{\omega}{1-\omega+\lambda\omega}}{1+\lambda_Y+\frac{\omega}{1-\omega+\lambda\omega}} \geq 1.
\end{aligned}$$

Note that $\hat{\delta} = 1$ if and only if $\omega = 1$. In this case (and only in this case) the political objective function admits time-consistent representation. In conclusion, the problem is isomorphic to the case of of one-sided altruism.

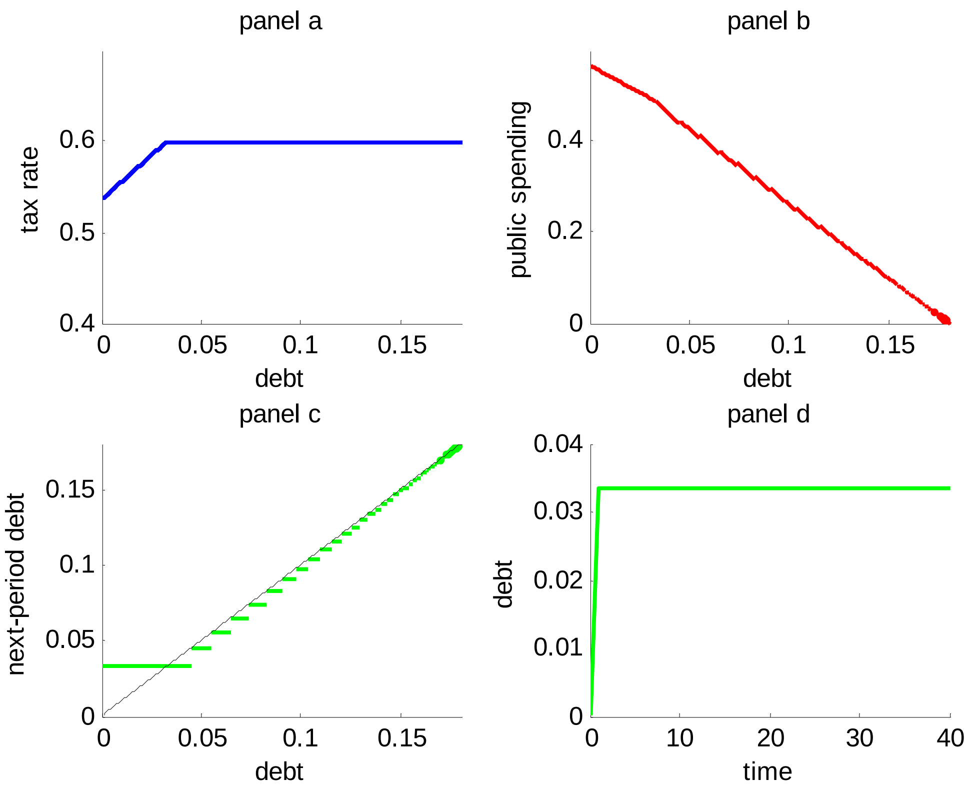
It is also useful to note that the preferences of the young continue to feature quasi-geometric discounting:

$$\begin{aligned} U_{Y,0} &= \lambda_Y \tilde{u}\left(c_O\left(A\left(\tau_{-1}\right)\right)\right) + \left(1+\lambda_Y\right) u\left(g_0\right) \\ &\quad + \left(\frac{1}{\lambda} + 1 + \lambda_Y\right) \sum_{t=1}^{\infty} \left(\lambda\beta\right)^t u\left(g_t\right) + \sum_{t=0}^{\infty} \left(\lambda\beta\right)^t \phi\left(A\left(\tau_t\right)\right), \end{aligned}$$

whereas the preferences of the old continue to be time consistent:

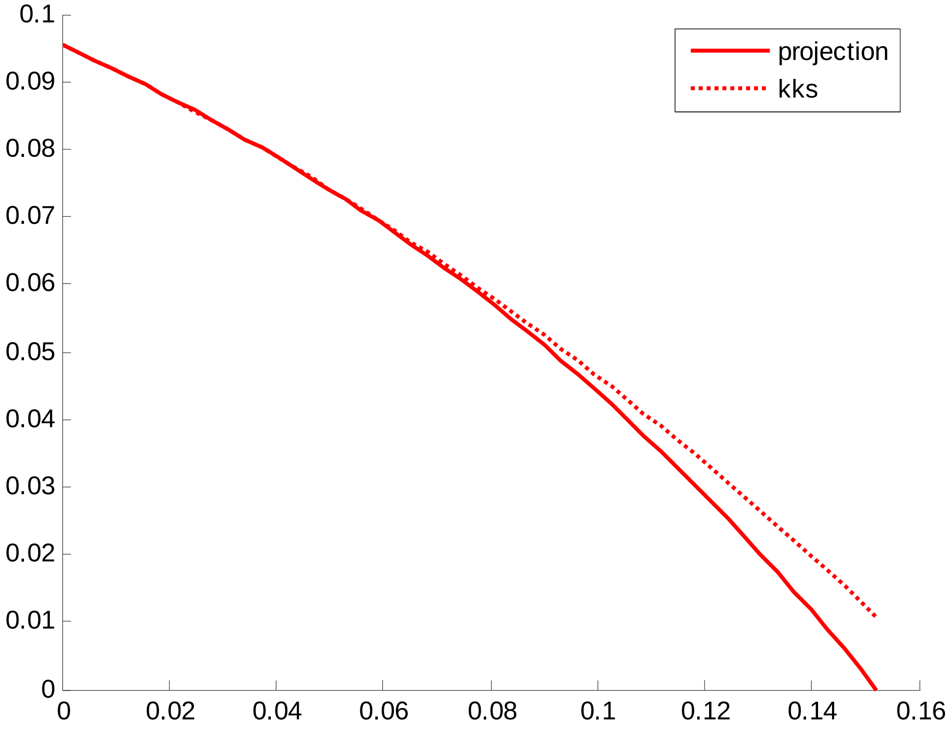
$$\begin{aligned} U_{O,0} &= \tilde{u}\left(c_O\left(A\left(\tau_{-1}\right)\right)\right) + \left(1+\lambda\left(1+\lambda_Y\right)\right) u\left(g_0\right) \\ &\quad + \sum_{t=1}^{\infty} \left(\lambda\beta\right)^t \left(1+\lambda\left(1+\lambda_Y\right)\right) u\left(g_t\right) + \sum_{t=0}^{\infty} \left(\lambda\beta\right)^t \lambda\phi\left(A\left(\tau_t\right)\right). \end{aligned}$$

Figure A1: Example II (ξ=1)



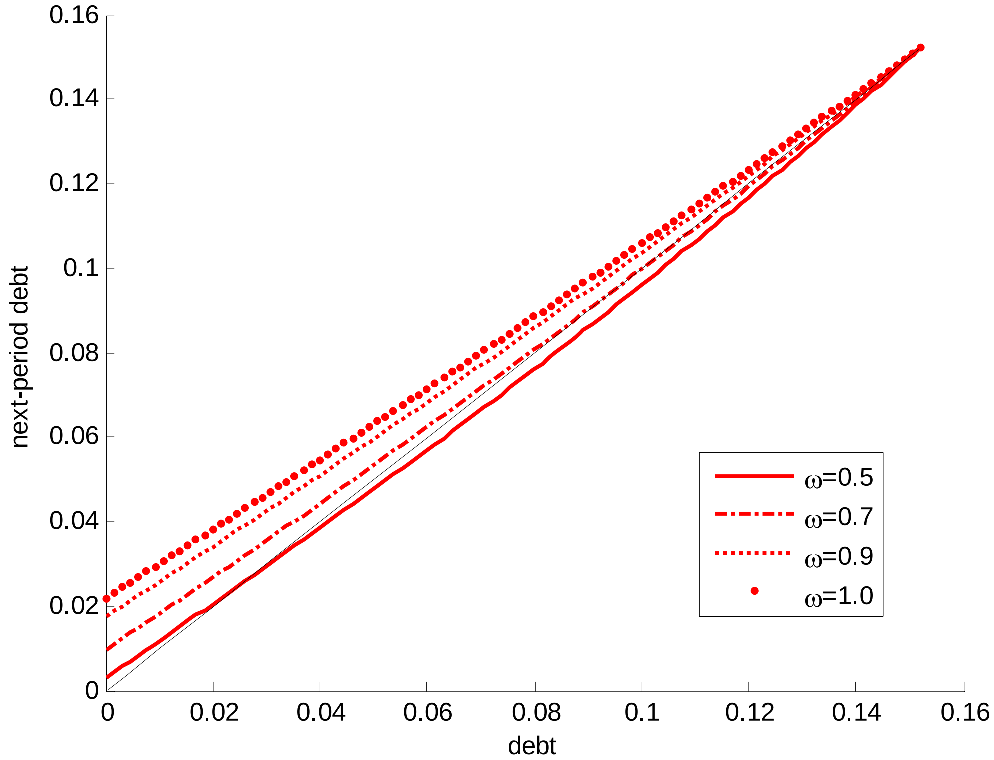
The figure shows equilibrium policy rules T(b) (panel a), G(b) (panel b), B(b) (panel c) and the equilibrium path of b (panel d). Parameter values are the same as in Figure 2.

Figure A2: The equilibrium policy rule G(b)



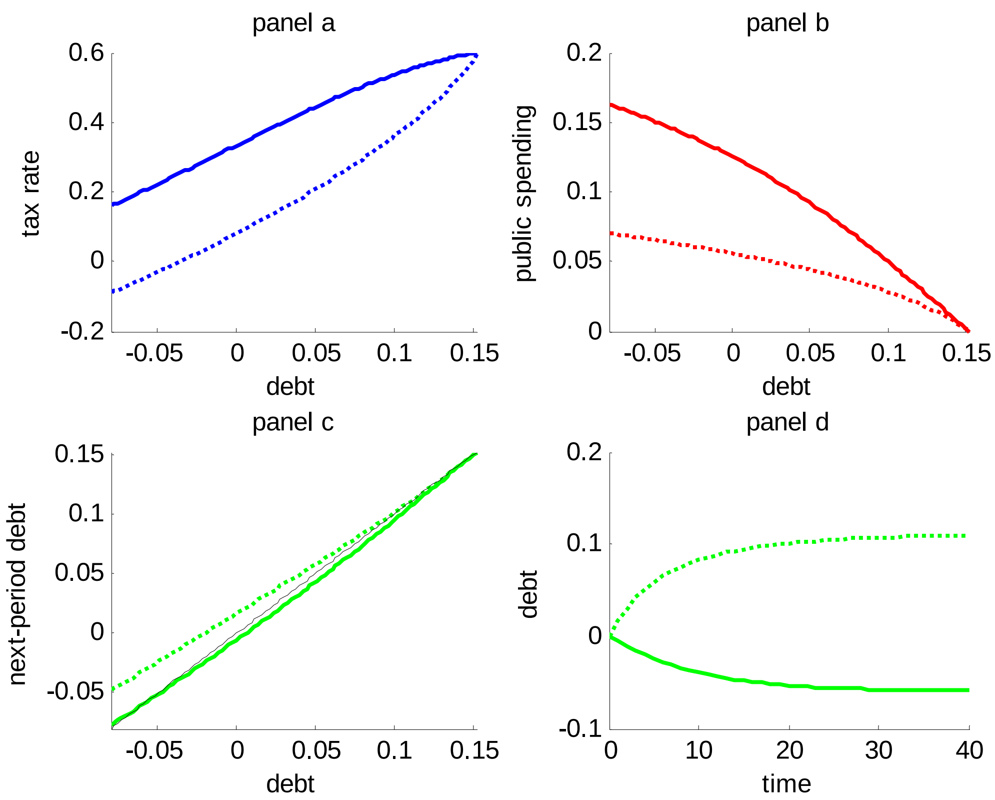
The figure shows the equilibrium policy rule G(b) solved by the projection method (solid line) and KKS method (dotted line), respectively. Parameter values are the same as in the calibrated economy (see Table 1).

Figure A3: The Equilibrium Policy Rule B(b) with different *ω*



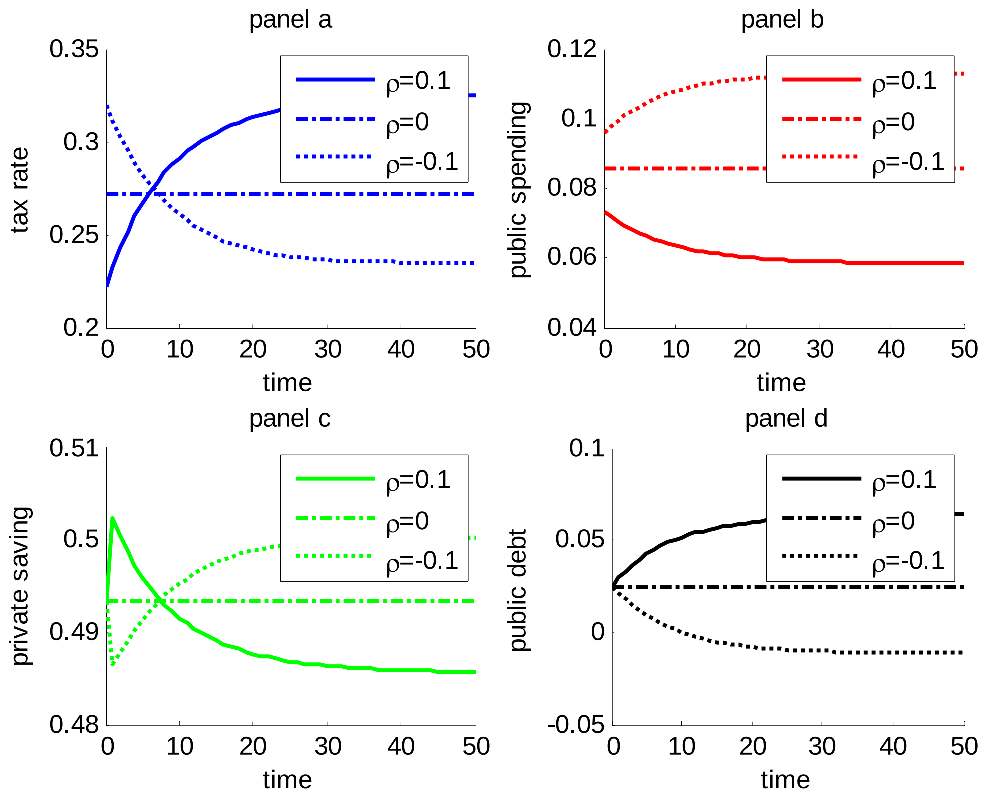
The figure shows the equilibrium policy rule B(b) with different values of ω. Otherwise, parameter values are the same as in the calibrated economy (see Table 1).

Figure A4: Different σ over Public Spending



The figure shows the equilibrium time paths of taxes (panel a), public spending (panel b), private savings (panel c) and debt (panel d) with different σ over public spending. The solid and dotted lines correspond to $\sigma = 1.25$ and $\sigma = 0.75$, respectively. Otherwise the parameter values are as in the benchmark calibration (see Table 1).

Figure A5: Non-Separable Preferences



The figure shows the equilibrium time paths of taxes (panel a), public spending (panel b), private savings (panel c) and debt (panel d), with non-separable preferences. The solid and dotted lines correspond to $\rho = 0.10$ (substitutes) and $\rho = -0.10$ (complements), respectively. The constant dashed lines correspond to $\rho = 0$ (separable utility). Otherwise the parameter values are as in the benchmark calibration (see Table 1). The initial conditions (b and s) are set equal to the steady state levels in the model with separable utility.