An Introduction to Social and Economic Networks

Lecture 3

Michael D. König

University of Zurich
Department of Economics

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Outline

Growing Networks

- Uniform Attachment
- Preferential Attachment
- Copying Models
- Entry and Exit

Uniform Attachment Networks

The network is constructed as follows:

- Time is measured at countable dates $t \geq 0$.
- A node that enters the network at time t is attached the label t.
- We initialize nodes $1, 2$ and the edge 12.
- Then, at every step $t > 2$ we add a new node t and create the edge ts, where node s is selected uniformly at random from the set $\{1, ..., t-1\}$ of already existing nodes in the network.

Degree Distribution

- Denote by $q_t(s,k) \in [0,1]$ the probability that a particular node s has degree k at time t where $s \leq t$.
- Any existing node s has degree $k \geq 1$ at time $t+1$ if, and only if, one of the following events occurs:

  (i) Node s had degree $k-1$ at time t (with probability $q_t(s,k-1)$) and receives a link by the entering node at time t (chosen uniformly at random with probability $\frac{1}{t+1}$), or

  (ii) node s already had degree k at time t (with probability $q_t(s,k)$) and is not chosen by the new node (with probability $1-\frac{1}{t+1}$).

- Thus we can write the time evolution of $q_{t+1}(s,k)$ according to the following *rate equation* (cf. "master equation")[1]

$$q_{t+1}(s,k) = \frac{1}{t+1} q_t(s,k-1) + \left(1 - \frac{1}{t+1}\right) q_t(s,k), \tag{1}$$

- The boundary conditions are[2]

$$\begin{aligned} q_1(0,k) &= q_1(1,k) = \delta_{k,1} \\ q_t(t,k) &= \delta_{k,1}. \end{aligned}$$

[1] Crispin W. Gardiner. *Handbook of Stochastic Methods: for Physics, Chemistry and the Natural Sciences.* Springer, Springer Series in Synergetics, 2004.

[2] The Kronecker-Delta is defined as $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

- Denote $p_t(k)$ the probability that a randomly selected node has any given degree k at time t. That is, $p_t(k)$ is the degree distribution at time t.
- Assuming that the selection of nodes is a sequence of stochastically independent events, it follows that

$$p_t(k) = \frac{1}{t+1} \sum_{s=0}^{t} q_t(s,k) \tag{2}$$

- Summation over all nodes $s = 0, ..., t$ in Equation (1) yields

$$\sum_{s=0}^{t} q_{t+1}(s,k) = \frac{1}{t+1} \sum_{s=0}^{t} q_t(s,k-1) + \left(1 - \frac{1}{t+1}\right) \sum_{s=0}^{t} q_t(s,k). \tag{3}$$

- Adding the term $q_{t+1}(t+1,k)$ on both sides gives

$$\begin{aligned}\sum_{s=0}^{t+1} q_{t+1}(s,k) &= \frac{1}{t+1}\sum_{s=0}^{t} q_t(s,k-1) \\ &\quad + \left(1-\frac{1}{t+1}\right)\sum_{s=0}^{t} q_t(s,k) + \delta_{k,1} \\ &= p_t(k-1) + tp_t(k) + \delta_{k,1}, \end{aligned} \tag{4}$$

- Note that we used the boundary condition $q_{t+1}(t+1,k) = \delta_{k,1}$. This reflects the fact that, in every period $t+1$, the entering node $t+1$ always represents a unit contribution to the set of nodes with degree 1 (and only these nodes).

- Then, with

$$(t+2)\frac{1}{t+2}\sum_{s=0}^{t+1} q_{t+1}(s,k) = (t+2)p_{t+1}(k), \tag{5}$$

we may write Equation (4) as follows

$$(t+2)p_{t+1}(k) - tp_t(k) = p_t(k-1) + \delta_{k,1}, \tag{6}$$

which is the law of motion of the degree distribution.

- In the limit $t \to \infty$, $p_t(k)$ attains its stationary distribution, denoted by $p(k)$, where

$$2p(k) = p(k-1) + \delta_{k,1}. \tag{7}$$

- We can solve the above equation for $k > 1$:

$$p(k) = 2^{-k}. \tag{8}$$

- Since there are no disconnected nodes in the network we have that $p(0) = 0$.
- For $k = 1$ we thus find that Equation (8) also solves Equation (7) for any $k = 1, 2, \ldots$. This means that the long run stationary degree distribution is geometric.

Preferential Attachment Networks

- The network is constructed in a similar way as in the uniform attachment network formation process.
- We initialize nodes $1, 2$ and edge 12, setting $t = 3$.
- Let $k_t(s)$ denote the degree of node s at time t.
- Then, at every step $t \geq 3$ we add a node t and create the edge ts with probability

$$k_t(s) / \sum_{r=0}^{t-1} k_t(r).$$

- The rate equation for the probabilities $q_t(s,k)$ that any node s has degree $k \geq 1$ at time t, $s \leq t$ is given by

$$q_{t+1}(s,k) = \frac{k-1}{2t} q_t(s,k-1) + \left(1 - \frac{k}{2t}\right) q_t(s,k). \tag{9}$$

- There are two exclusive events that may lead node s to have degree k in time step $t+1$:

  (i) Node s had degree $k-1$ at time t and the new node $t+1$ establishes a link to s, or

  (ii) node s had degree k at time t and the new node $t+1$ does not form a link to it.

- The probability of event (i) is given by $q_t(s, k-1)$ multiplied by the ratio of the degree, $k-1$, to the sum of the degrees, that is $2t$.
- The probability of the event (ii) is the complement of the probability that the new node establishes a link to s with degree k, that is $1 - \frac{k}{2t}$ times $q_t(s, k)$.
- Summing over all nodes $s \leq t+1$ in Equation (9) and adding the term $q_{t+1}(t+1, k)$ on both sides, we arrive at

$$\sum_{s=0}^{t+1} q_{t+1}(s, k) = \frac{k-1}{2t} \sum_{s=0}^{t} q_t(s, k-1) + \left(1 + \frac{k}{2t}\right) \sum_{s=0}^{t} q_t(s, k) + \delta_{k,1}.$$

- We have that

$$\sum_{s=0}^{t+1} q_{t+1}(s,k) = \frac{1}{2}\frac{t+1}{t}\left[(k-1)\frac{1}{t+1}\sum_{s=0}^{t} q_t(s,k-1)\right.$$
$$\left.-k\frac{1}{t+1}\sum_{s=0}^{t} q_t(s,k)\right]$$
$$+(t+1)\frac{1}{t+1}\sum_{s=0}^{t} q_t(s,k) + \delta_{k,1}$$
$$= \frac{1}{2}\frac{t+1}{t}\left((k-1p_t(k-1) - kp_t(k))\right)$$
$$+(t+1)p_t(k) + \delta_{k,1}.$$

- Using the fact that

$$\sum_{s=0}^{t+1} q_{t+1}(s,k) = (t+2)\frac{1}{t+2}\sum_{s=0}^{t+1} q_{t+1}(s,k)$$
$$= (t+2)p_{t+1}(k),$$

- we get

$$(t+2)p_t(k) = \frac{1}{2}\frac{t+1}{t}\left((k-1)p_t(k-1) - kp_t(k)\right)$$
$$(t+1)p_t(k) + \delta_{k,1}.$$

- In the limit, as $t \to \infty$, $p_t(k)$ converges to its stationary distribution, $p(k)$, and we can write

$$p(k) = \frac{1}{2}\left((k-1)p(k-1) - kp(k)\right) + \delta_{k,1}, \tag{10}$$

since $p_{t+1}(k) = p_t(k)$ in the stationary state and for large t, $t+2 \sim t+1 \sim t$.

- The solution for $k > 1$ of Equation (10) is given by

$$p(k) = \frac{4}{k(k+1)(k+2)} \sim k^{-3}. \tag{11}$$

- Alternatively, one can write Equation (10) in the form

$$\begin{aligned} p(k) &= \frac{1}{2}\left(k\left[p(k-1) - p(k)\right] - p(k-1)\right) + \delta_{k,1} \\ &= -\frac{1}{2}\left(k\frac{p(k) - p(k-\Delta k)}{\Delta k} + p(k-\Delta k)\right) + \Delta_k, \end{aligned}$$

where $\Delta k = 1$.

- Taking the limit $\Delta k \to 0$ (aka the "continuum approximation")[3] one obtains the continuous form of Eq. (10)

$$\begin{aligned} p(k) &= -\tfrac{1}{2}\left(k\tfrac{dp}{dk} + p(k)\right) \\ &= -\tfrac{1}{2}\tfrac{d}{dk}\left(kp(k)\right). \end{aligned} \tag{12}$$

- The solution of this equation is given by

$$p(k) = 2k^{-3}, \tag{13}$$

  where the factor 2 comes from the normalization condition $\int_1^\infty p(k)dk = 1$.
- We find, therefore, that the degree distribution satisfies a power law of the form $p(k) \propto k^{-\gamma}$ with exponent $\gamma = 3$.

[3] S.N. Dorogovtsev and J.F.F. Mendes. "Evolution of networks". *Advances in Physics* 51.4 (2002), pp. 1079–1187. ISSN: 0001-8732.

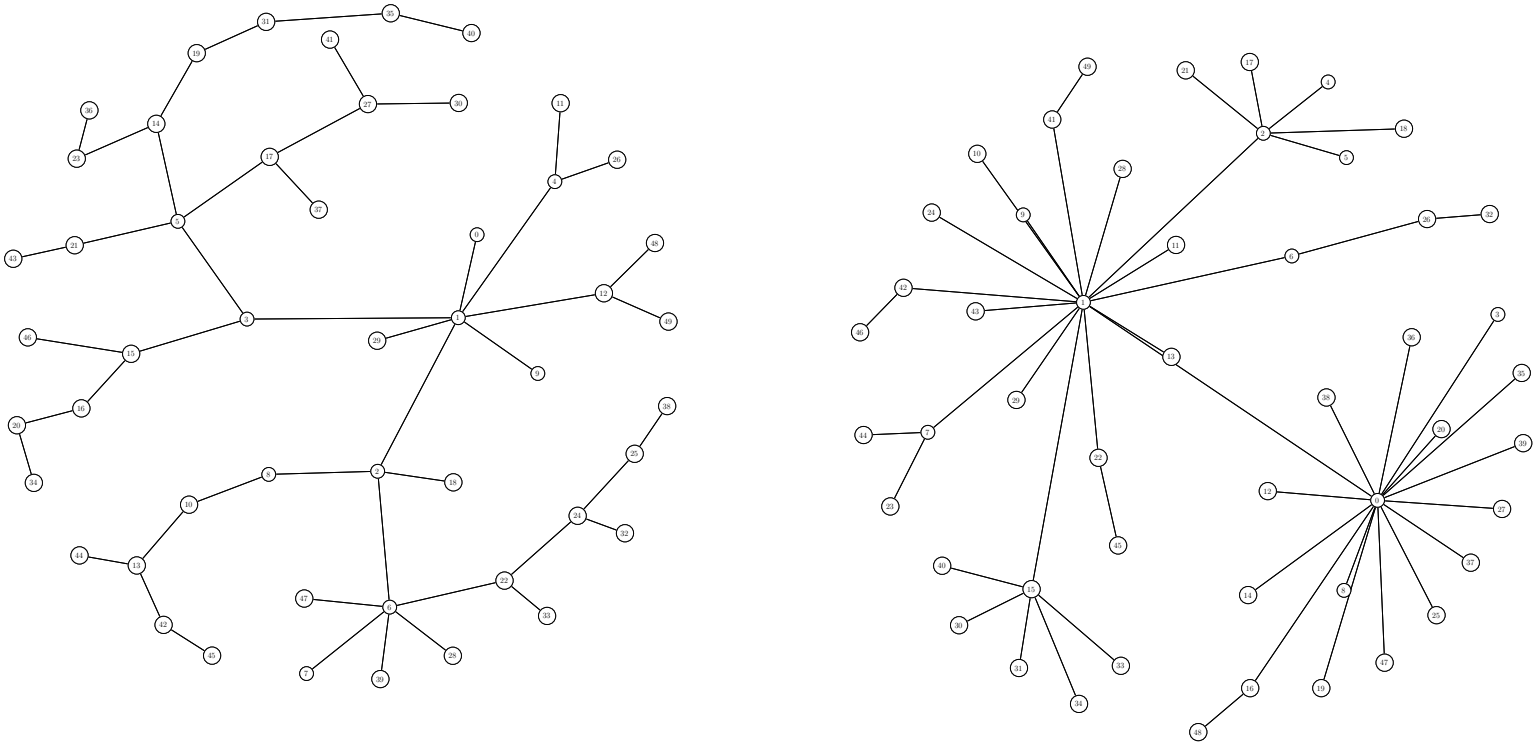

Figure: Uniform attachment (left) and preferential attachment (right) networks with $n = 50$ nodes.

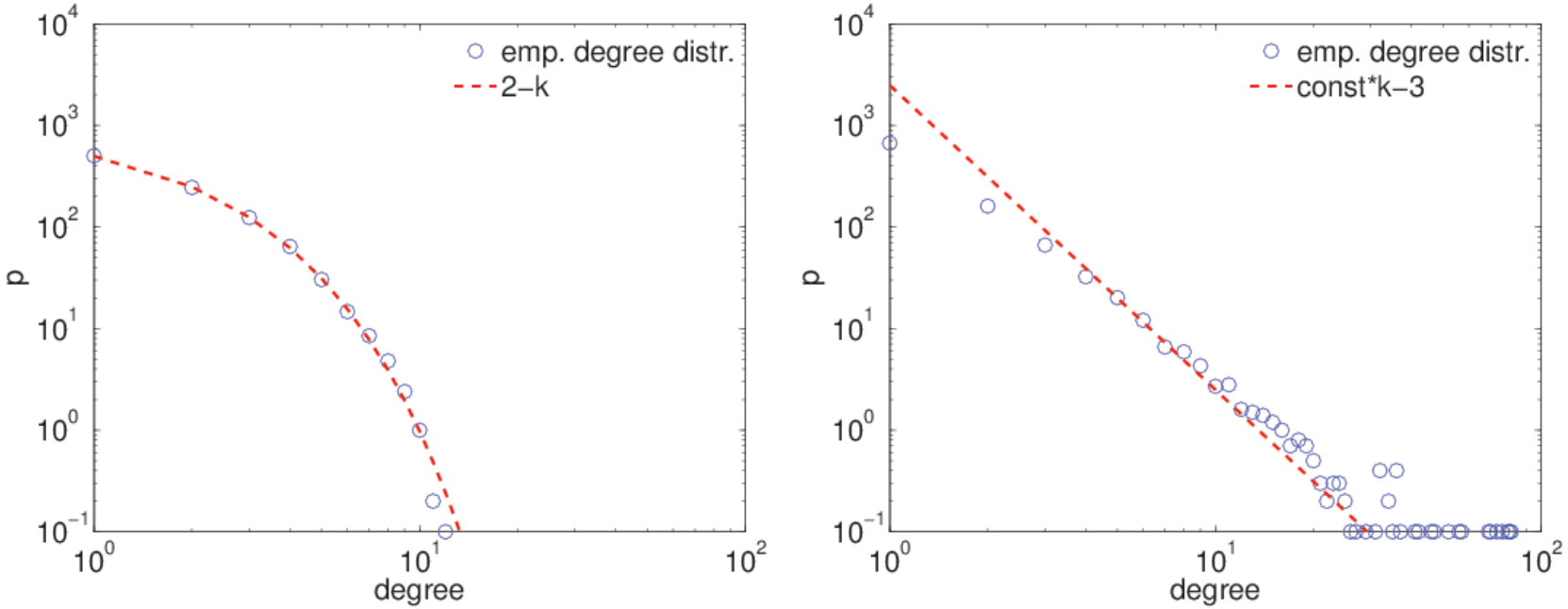

Figure: Degree distribution of the uniform (left) and preferential attachment (right) networks for $n = 1000$ averaged over 10 realizations.

Copying Models[4]

- For a given network $G = (\mathcal{N}, \mathcal{E}) \in \mathcal{G}^n$ we assign each agent $i \in \mathcal{N}$ a payoff $\pi_i(\cdot, \delta) : \mathcal{G}^n \to \mathbb{R}$ which depends on the network G and a (decay) parameter $\delta \geq 0$ which measures the degree of interdependency between agents' payoffs in G.
- We define the *link incentive function* $f_i : \mathcal{G}^n \times \mathcal{N} \to \mathbb{R}$ for an agent $i \in \mathcal{N}$ as

$$f_i(G, j) \equiv \pi_i(G + ij, \delta) - \pi_i(G, \delta), \tag{14}$$

  which measures the marginal payoff to the agent i resulting from the potential link $ij \notin \mathcal{E}$.

[4] Michael D. König. "The Formation of Networks with Local Spillovers and Limited Observability". *Theoretical Economics* 11 (2016), pp. 813–863.

- Here we focus on link incentive functions (and therefore on classes of games) which satisfy the following conditions:
- **Assumption:** For all $i \in \mathcal{N}$ the link incentive function $f_i(G, \cdot) : \mathcal{N} \to \mathbb{R}$ has the following properties

(LM) Link monotonicity: $f_i(G, j) \geq 0$ for all $j \neq i \in \mathcal{N}$.

(LD) Linear differences: For all $ij, ik \notin \mathcal{E}$, there exists a constant $\gamma \geq 0$ and a linear increasing function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{f_i(G, j) - f_i(G, k)}{\delta^\gamma} = g\left(d_j - d_k\right) + o(1),$$

holds in the limit of $\delta \to 0$.

- Before creating links, an entering agent t must make an observation of the prevailing network G_{t-1} and identify a set of agents to whom he can form links. We call this set the (observed) sample $\mathcal{S}_t \subseteq \mathcal{P}_{t-1}$.
- The sample $\mathcal{S}_t$ is obtained by selecting $n_s \geq 1$ incumbent nodes in $\mathcal{P}_{t-1}$ uniformly at random (without replacement) and forming the union of these nodes and their out-neighbors. We call n_s the *observation radius*.
- We assume that an entrant t chooses to link to the incumbent node $j \in \mathcal{S}_t$ that maximizes the value of his link incentive function plus a random element[5]

$$f_t(G_{t-1}, j) + \varepsilon_{ij}. \tag{15}$$

where the term ε_{ij} is an exogenous random variable.

[5] Tom A.B. Snijders. "The Statistical Evaluation of Social Network Dynamics". *Sociological Methodology* 31.1 (2001), pp. 361–395; Tom A.B. Snijders, Johan Koskinen, and Michael Schweinberger. "Maximum likelihood estimation for social network dynamics". *The Annals of Applied Statistics* 4.2 (2010), pp. 567–588.

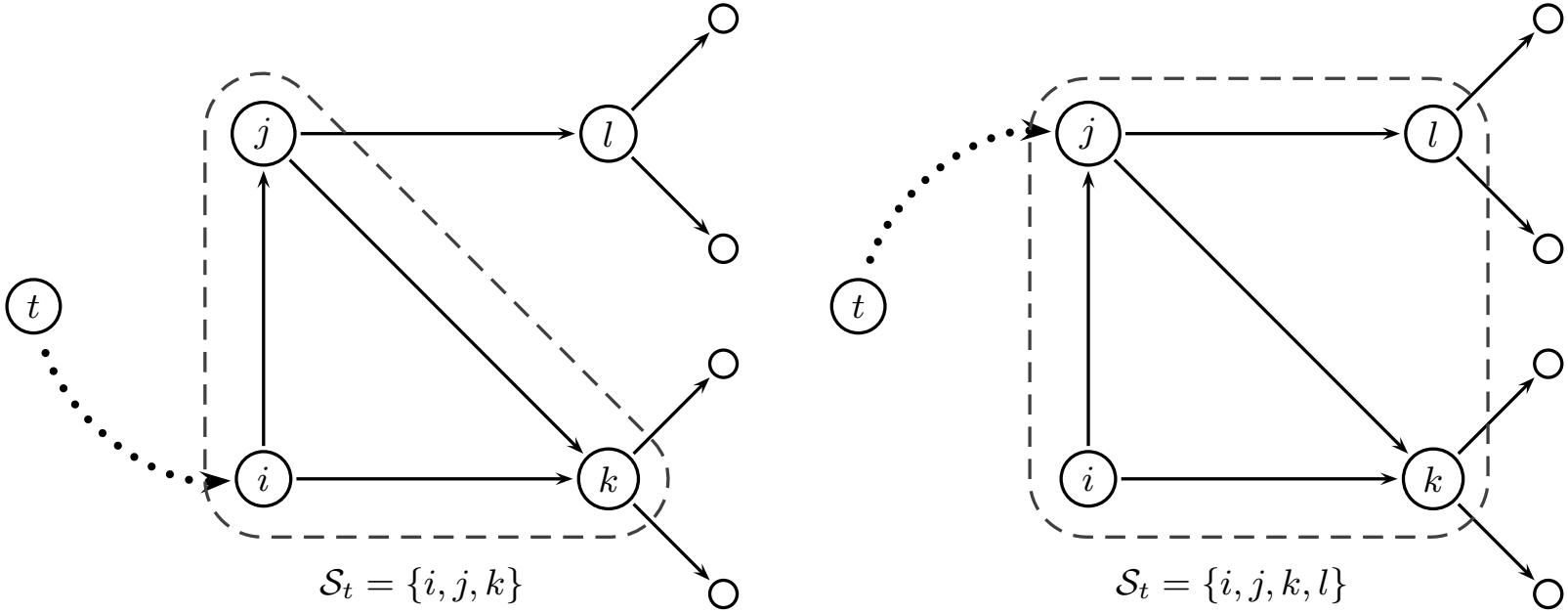

Figure: Illustration of the network sampling procedure. (Left panel) In the first draw, the entering agent t observes agent i and its out-neighbors j, k. The observed sample is $\mathcal{S}_t = \{i, j, k\}$. (Right panel) In the second draw, agent t observes also agent j and the out-neighborhood $\{k, l\}$ of j. The observed sample is then $\mathcal{S}_t = \{i, j, k, l\}$.

- **Definition:** For a fixed $T \in \mathbb{N} \cup \{\infty\}$ we define a network formation process $(G_t)_{t \in [T]}$, $[T] \equiv \{1, 2, \ldots, T\}$, as follows. Given the initial graph $G_1 = \ldots = G_{m+1} = K_{m+1}$, for all $t > m + 1$ the graph G_t is obtained from G_{t-1} by applying the following steps:

  (i) **Growth:** Given $\mathcal{P}_1$ and $\mathcal{A}_1$, for all $t \geq 2$ the agent sets in period t are given by $\mathcal{P}_t = \mathcal{P}_{t-1} \cup \{t\}$ and $\mathcal{A}_t = \mathcal{A}_{t-1} \setminus \{t\}$, respectively.

  (ii) **Network sampling:** Agent t observes a sample $\mathcal{S}_t \subseteq \mathcal{P}_{t-1}$. The sample S_t is constructed by selecting $n_s \geq 1$ agents $i \in \mathcal{P}_{t-1}$ uniformly at random without replacement and adding i as well as the out-neighbors $\mathcal{N}^+_{G_{t-1}}(i)$ of i to $\mathcal{S}_t$.

  (iii) **Link creation:** Given the sample $\mathcal{S}_t$, agent t creates $m \geq 1$ links to agents in $\mathcal{S}_t$ without replacement. For each link, agent t chooses the $j \in \mathcal{S}_t$ that maximizes $f_t(G_{t-1}, j) + \varepsilon_{tj}$.

- We next define the *attachment kernel* as the probability that an agent $j \in \mathcal{P}_{t-1}$ receives a link from the entrant. It can be written as

$$K_t^{\beta}(j|G_{t-1}) = \sum_{\mathcal{S}_t \subseteq \mathcal{P}_{t-1}} K_t^{\beta}(j|\mathcal{S}_t, G_{t-1}) \mathbb{P}_t(\mathcal{S}_t|G_{t-1}),$$

where $K_t^{\beta}(j|\mathcal{S}_t, G_{t-1})$ is the probability, conditional on the sample $\mathcal{S}_t$ and the prevailing network G_{t-1}, that an agent j receives a link after the m draws (without replacement) by the entrant.

- Assume that the exogenous random terms ε_{tj} are identically and independently type I extreme value distributed (or Gumbel distributed) with parameter η.[6]
  Then the probability that an entering agent t chooses the passive agent $j \in \mathcal{S}_t$ for creating the link tj (in the first of the m draws of link creation) follows a *multinomial logit* distribution given by[7]

$$\mathbb{P}_t\left(f_t(G_{t-1}, j) + \varepsilon_{tj} = \max_{k \in \mathcal{S}_t} f_t(G_{t-1}, k) + \varepsilon_{tk}\right) \approx \frac{e^{\beta d_j(G_{t-1})}}{\sum_{k \in \mathcal{S}_t} e^{\beta d_k(G_{t-1})}}, \tag{16}$$

  where we have applied condition (LD) for the link incentive function $f_t(G_{t-1}, \cdot)$, dropped terms of the order $o(\delta^b)$ and denoted by $\beta \equiv \eta\delta^b$.

[6] For a type-I extreme value distributed random variable ε we have that $\mathbb{P}(\varepsilon \leq c) = e^{-e^{c/\zeta - \gamma}}$, where $\gamma \approx 0.58$ is Euler's constant. The mean is $\mathbb{E}(\varepsilon) = 0$ and the variance is given by $\mathrm{Var}(\varepsilon) = \frac{\pi^2\zeta^2}{6}$.

[7] Simon P Anderson, Andre De Palma, and Jacques François Thisse. *Discrete choice theory of product differentiation.* MIT Press, 1992.

The Emergence of Quasi-Stars

- A *quasi-star* S_n^m, $n \geq m+1$, with node set $[n] \equiv \{1, \ldots, n\}$ is a directed graph in which all nodes in the set $[m+1]$ in S_n^m are bilaterally connected, while the nodes in the set $[n-1] \backslash [m+1]$ all maintain an outgoing link to the agents in the set $[m]$.

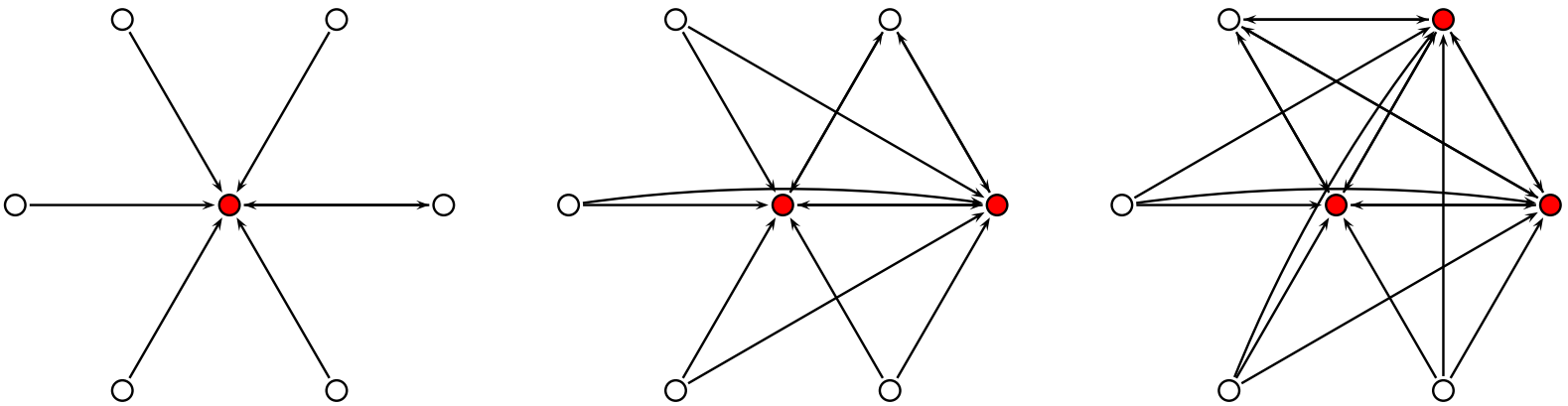

Figure: Illustration of the quasi-stars S_7^1, S_7^2 and S_7^3. Filled circles indicate the nodes with the highest degree.

Proposition: Let $(G_t^\beta)_{t\in[T]}$ be a sequence of networks generated with observation radius $n_s^{(1)}$, and $(H_t^\beta)_{t\in[T]}$ be a sequence of networks generated with observation radius $n_s^{(2)}$ such that $n_s^{(1)} > n_s^{(2)}$. Let $\Sigma_T^m \subset \mathcal{G}^T$ be the class of *quasi-stars* of order $T > m+1$. Then,

(i) in the limit of vanishing noise, we have that

$$\lim_{\beta\to\infty} \mathbb{P}(H_T^\beta \in \Sigma_T^m) = \mathbb{P}(G_T^\beta \in \Sigma_T^m) = 1;$$

(ii) in the limit of strong noise, we have that

$$\lim_{\beta\to 0} \mathbb{P}(H_T^\beta \in \Sigma_T^m) > \mathbb{P}(G_T^\beta \in \Sigma_T^m) > 0.$$

Large Noise Limit and the Distribution of Degree

- We restrict our discussion to the case of "strong noise" when $\beta = 0$. In this case we have that the attachment kernel (which gives the probability that j receives a link from the entering agent given that j is in the sample $\mathcal{S}_t$) is

$$K_t^0(j|\mathcal{S}_t, G_{t-1}) = \frac{m}{|\mathcal{S}_t|}\mathbb{1}_{\mathcal{S}_t}(j).$$

- The sample size is bounded by $|\mathcal{S}_t| \leq n_s(m+1)$. If no agent enters the sample more than once, then equality holds. The sample $\mathcal{S}_t$ is constructed by selecting n_s nodes from $\mathcal{P}_{t-1}$ without replacement, and forming the union of these nodes and their out-neighbors.
- Assuming that $n_s = o(t)$ and $d_{G_{t-1}}(j) = o_p(t)$, the probability that a node is entering $\mathcal{S}_t$ more than once is of the order $o(t)$ and thus

$$\frac{1}{|\mathcal{S}_t|} = \frac{1}{n_s(m+1)} + o_p\left(\frac{1}{t}\right). \tag{17}$$

- The unconditional probability that an agent $j \in \mathcal{P}_{t-1}$ receives a link by the entrant t is then given by

$$K_t^0(j|G_{t-1}) = \tfrac{1}{n_s(m+1)}\mathbb{P}(j \in \mathcal{S}_t|G_{t-1}) + o\left(\tfrac{1}{t}\right).$$

- If the degree of node j is small compared to the network size t, i.e. $d_{G_{t-1}}(j) = o_p(t)$, and the observation radius is small such that $n_s = o(t)$, then $\mathbb{P}(j \in \mathcal{S}_t|G_{t-1}) = n_s \frac{1+d_{G_{t-1}}(j)}{t} + o\left(\frac{1}{t}\right)$, and we obtain

$$K_t^0(j|G_{t-1}) = \frac{1}{1+m}\frac{1+d_{G_{t-1}}(j)}{t} + o\left(\frac{1}{t}\right). \tag{18}$$

Degree Distribution

- We can state the following result for the asymptotic degree distribution when the *observation radius is small*.
- **Proposition:** Consider the sequence of degree distributions $\{P_t\}_{t\in\mathbb{N}}$ generated by an indefinite iteration of the network formation process $(G_t^\beta)_{t\in\mathbb{N}}$ with a small observation radius $n_s = o(t)$. Assume that $\beta = 0$ and $d_{G_{t-1}}(j) = o_p(t)$ for all $j \in \mathcal{P}_{t-1}$. Then, we have that $P_t(k) \to P(k)$, almost surely, where

$$P(k) = k^{-\left(2+\frac{1}{m}\right)}\left(1 + O\left(\frac{1}{k}\right)\right), \tag{19}$$

for all $k \geq 0$ as $t \to \infty$.

- **Proof:** For all $t \geq 1$ we denote by $N_t(k) \equiv \sum_{i=0}^{t} \mathbb{1}_k(d_{G_t}(i))$ the number of nodes in the graph G_t with in-degree k.
- The relative frequency of nodes with in-degree k is accordingly defined as $P_t^\beta(k) \equiv \frac{1}{t} N_t(k)$ for all $t \geq 1$. The sequence $\{P_t^\beta(k)\}_{k \in \mathbb{N}}$ is the (empirical) *degree distribution*.
- We will now derive a recursive system which can be used to describe the time evolution of the expected degree distribution. Let $N_t \equiv \{N_t(k)\}_{k \geq 0}$. Denoting by $k = d^-_{G_{t-1}}(j)$ we write the attachment kernel as

$$K_t^\beta(j|G_{t-1}) = \frac{a(k)}{t\zeta(\beta, m)} + o\left(\frac{1}{t}\right).$$

- The expected number of nodes with in-degree k at time t can increase by the creation of a link to a node with in-degree $k-1$, or it decreases by the creation of a link to a node with in-degree k.
- It then follows that

$$\mathbb{E}[N_{t+1}(k)|N_t] = N_t(k)\left(1 - \frac{a(k)}{t\zeta(\beta,m)}\right) + N_t(k-1)\frac{a(k-1)}{t\zeta(\beta,m)} + \delta_{0,k} + o\left(\frac{1}{t}\right). \tag{20}$$

- Taking expectations on both sides of Equation (20), dividing by $t+1$, and denoting by $P_t^\beta(k) = \mathbb{E}[N_t(k)]$, gives us

$$P_{t+1}^\beta(k) = \frac{t}{t+1}\left[P_t^\beta(k)\left(1 - \frac{a(k)}{t\zeta(\beta,m)}\right) + P_t^\beta(k-1)\frac{a(k-1)}{t\zeta(\beta,m)} + \frac{1}{t}\delta_{0,k}\right] + o\left(\frac{1}{t}\right).$$

- Some algebraic manipulations allow us to write this as

$$P_{t+1}^{\beta}(k) - P_t^{\beta}(k) = b_t(k)\left[c_t(k) - P_t^{\beta}(k)\right] + o\left(\frac{1}{t}\right), \tag{21}$$

where

$$b_t(k) \equiv \frac{\zeta(\beta, m) + a(k)}{\zeta(\beta, m)} \frac{1}{t+1},$$

$$c_t(k) \equiv P_t^{\beta}(k-1) \frac{a(k-1)}{\zeta(\beta, m) + a(k)} + \frac{\zeta(\beta, m)}{\zeta(\beta, m) + a(k)} \delta_{0,k}.$$

- One can show that the solution of this recursive expression is given by[8]

$$P^{\beta}(k) = \frac{\zeta(\beta, m)}{\zeta(\beta, m) + a(0)} \prod_{j=1}^{k} \frac{a(j-1)}{\zeta(\beta, m) + a(j)}. \tag{22}$$

[8] Michael D. König. "The Formation of Networks with Local Spillovers and Limited Observability". *Theoretical Economics* 11 (2016), pp. 813–863.

- For $\beta \to 0$ the attachment kernel is given by

$$K_t^{\beta}(j|G_{t-1}) = \frac{a(k)}{t\zeta(\beta, m)} + o\left(\frac{1}{t}\right),$$

where $k = d_{G_{t-1}}(j)$, $a(k) = 1 + \beta k$ and $\zeta(\beta, m) = \frac{1+\beta m}{m}$.

- We then can apply Equation (22), noting that the product on the right-hand side admits a closed-form representation in terms of Gamma functions as

$$P^{\beta}(k) = \frac{1+\beta m}{1+m(1+\beta)} \frac{\Gamma\left(\frac{1}{\beta}+k\right)\Gamma\left(2+\frac{1+\beta m}{\beta m}\right)}{\Gamma\left(\frac{1}{\beta}\right)\Gamma\left(2+\frac{1+m}{1+\beta m}+k\right)}. \tag{23}$$

- By *Stirling's formula* we can approximate the Gamma function for large k as[9]

$$\frac{\Gamma(k)}{\Gamma(k+c)} = k^{-c}\left(1 + O\left(\frac{1}{k}\right)\right). \tag{24}$$

- For the tails of the degree distribution in Equation (23) this implies that

$$P^{\beta}(k) \sim (1+\beta k)^{-(2+\frac{1}{\beta m})}\left(1 + O\left(\frac{1}{k}\right)\right)$$

for large k.

[9] By Stirling's formula we can approximate the Gamma function for large k as $\Gamma(k) = \sqrt{\frac{2\pi}{k}}\left(\frac{k}{e}\right)^k\left(1 + O\left(\frac{1}{k}\right)\right)$. Hence, $\frac{\Gamma(k)}{\Gamma(k+a)} = \left(1 + O\left(\frac{1}{k}\right)\right)\sqrt{(1+a/k)}(1+a/k)^{-k}\left(\frac{k}{k+a}\right)^k\left(\frac{k+a}{e}\right)^{-a}$. Since $\sqrt{(1+a/k)} \to 1$ for $k \to \infty$ this term is asymptotically negligible. Additionally $(1+a/k)^{-k} \to e^{-a}$ for $k \to \infty$, and $(k+a)^{-a} \sim k^{-a}$ for $k \to \infty$. Therefore, the leading order approximation of the ratio of Gamma functions is given by $\frac{\Gamma(k)}{\Gamma(k+a)} = k^{-a}\left(1 + O\left(\frac{1}{k}\right)\right)$.

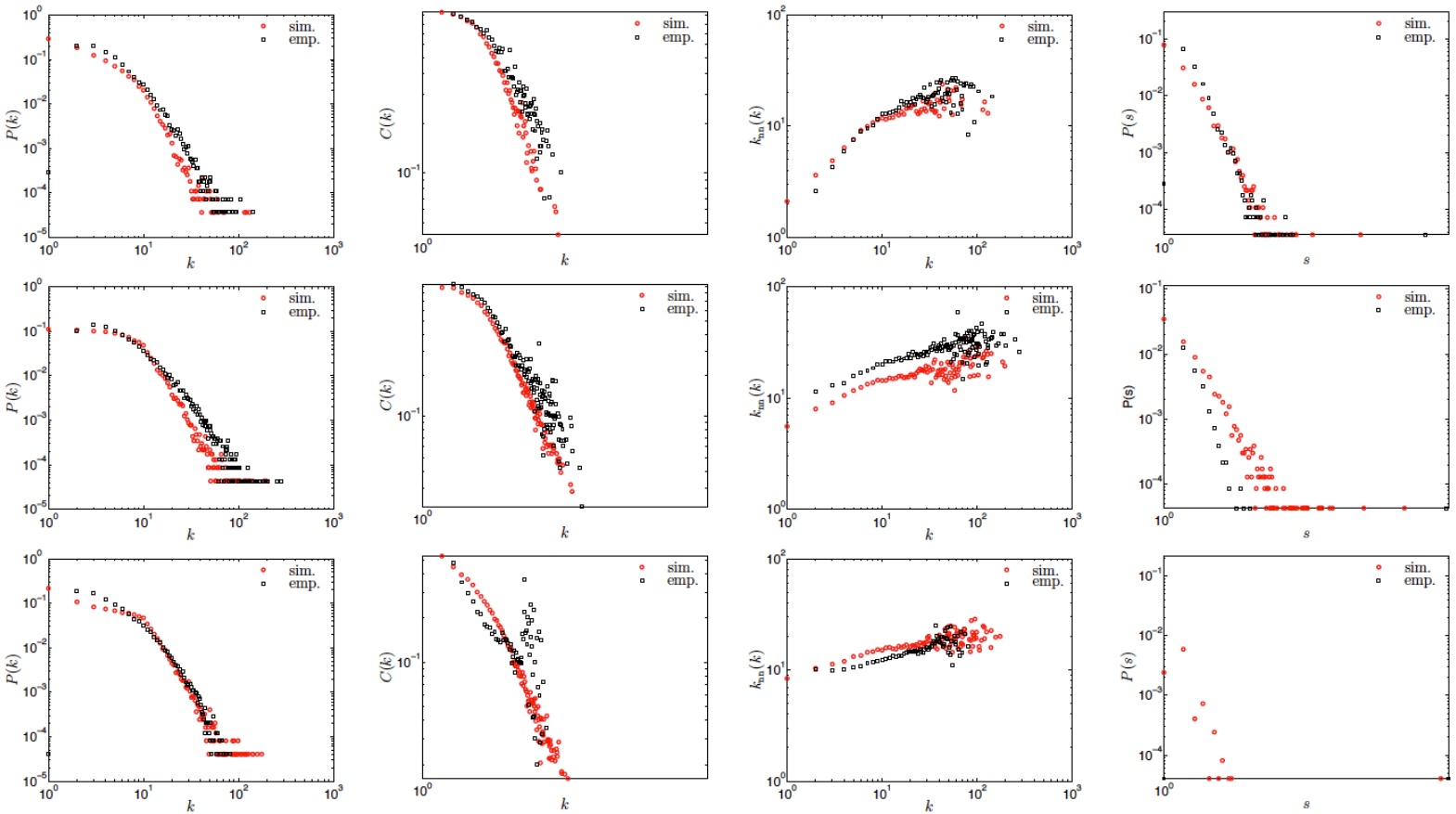

Figure: Empirical degree distribution $P(d)$ (first column), clustering-degree correlation $C(d)$ (second column), average nearest neighbor connectivity $k_{\mathrm{nn}}(d)$ (third column) and component size distribution $P(s)$ (fourth column) constructed from (first row) USPTO patents on drugs, (second row) coauthors in condensed matter physics from the arXiv database, and (third row) the network of coauthors in economics from the CollEc database (empirical data points indicated by □) and the corresponding distributions generated by the model (indicated by ○).

Entry and Exit

- In the following we introduce an endogenous network formation model[10] in which firms exit due to
  - exogenous *shocks*, or the
  - *propagation* of shocks through the network.
- Firms can replace suppliers they have lost due to exit subject to *switching costs* and search frictions (rewiring).
- This enables us to study the impact of shocks on aggregate production in an adaptive network.
- We show that depending on the nature of the shocks, *adaptivity* can make the network *more or less stable*.

[10] Michael D. König. "Aggregate Fluctuations in Adaptive Production Networks". *University of Zurich, Department of Economics Working Paper No. 240* (2016).

Definition: We consider a Markov chain $(G_t)_{t=0}^{\infty}$ comprising a sequence of networks $G_0, G_1, \ldots$ where $G_t = (\mathcal{N}_t, \mathcal{E}_t)$ with $\mathcal{N}_t$ being the set of firms and $\mathcal{E}_t$ the set of (directed) edges/links between them. Starting from an initial state G_0, in every step from t to $t+1$ the following events happen:

- **Entry:** At every period $t = 1, 2, \ldots,$ a new firm i is born and selects a randomly chosen firm j among the incumbents as a supplier.
- **Exit:**
  - (i) Large Shocks: With probability $\delta \in [0, 1]$ a randomly selected firm exits.
  - (ii) Small Shocks: With probability $r \in [0, 1]$ a randomly selected firm is hit a shock and exits with probability $\mathbb{P}(i \text{ exits}) = \frac{1}{1+d_i^+}$.
  - (iii) Shock Propagation: With probability $\rho \in [0, 1]$ a firm which does not have a supplier exits.
- **Rewiring:** If a firm looses a supplier due to exit, with probability $\gamma \in [0, 1]$ it attempts to replace it with a firm drawn uniformly at random among the incumbents.

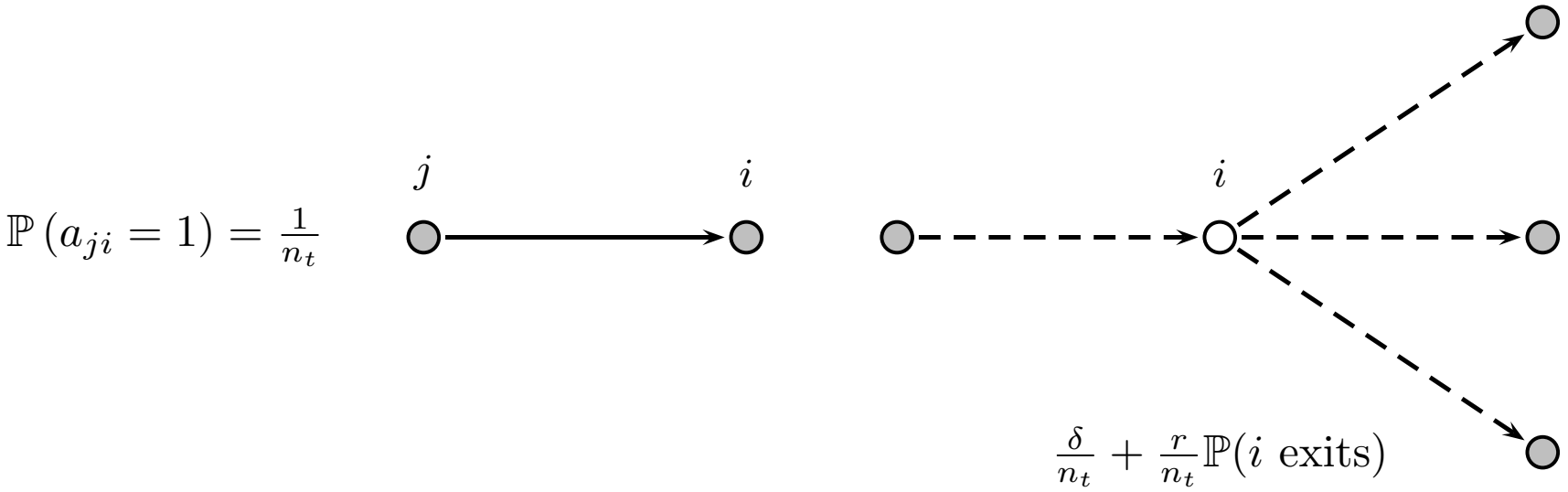

Figure: (Left panel) Entry of a firm i and uniform attachment to an incumbent firm j with probability $\mathbb{P}\left(a_{ji}=1\right)=\frac{1}{n_t}$. (Right panel) The exit of a firm i due to a large shock with probability $\frac{\delta}{n_t}$ or small shock with probability $\frac{r}{n_t}\mathbb{P}(i \text{ exits})=\frac{r}{n_t}\frac{1}{1+d_i^+}$. Filled circles indicate firms that have not exited, while empty circles indicate firms that have exited.

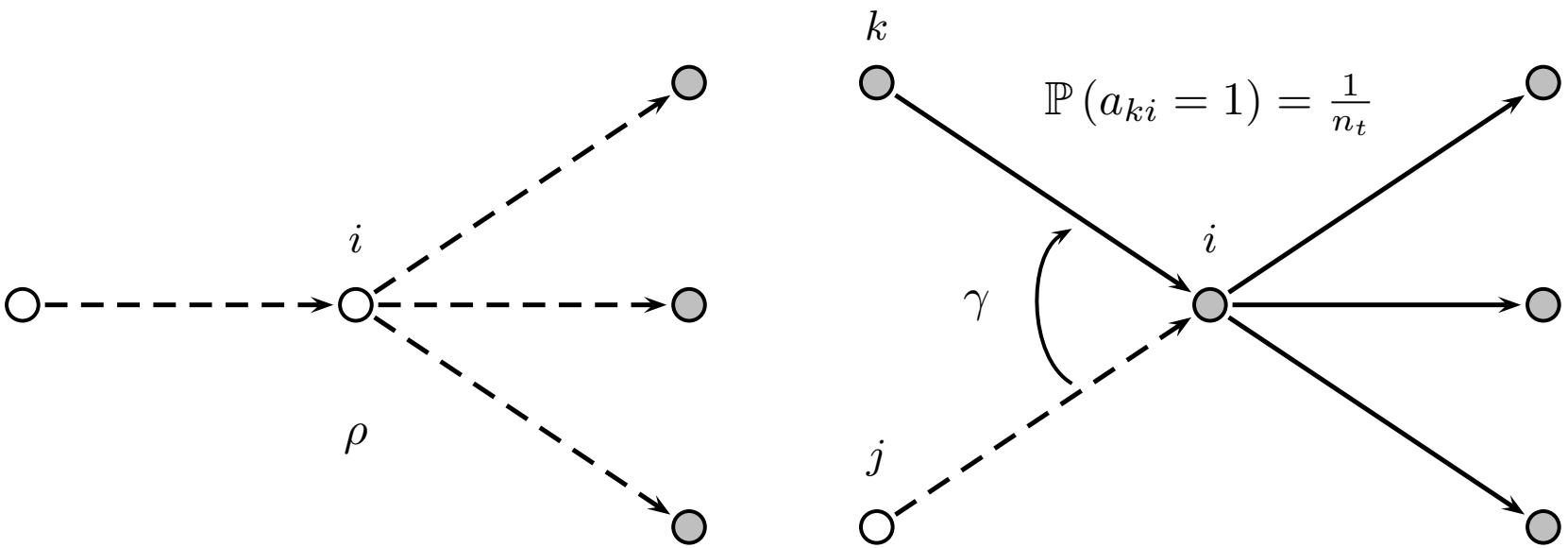

Figure: (Left panel) Shock propagation leading to the exit of a firm i with in-degree zero with probability ρ. (Right panel) Replacement of a supplier j that has exited with probability γ. Then a new firm k becomes the supplier to i with probability $\mathbb{P}(a_{ki} = 1) = \frac{1}{n_t}$.

- As larger firms have a lower probability to exit, we call this *preferential survival*.[11] This is consistent with the empirical evidence.[12]
- The parameter γ is a measure of the *adaptivity* of the network.
- It is similar to macroeconomic models of price stickiness, where the opportunity for firms to reset their prices in any particular period is a random event, and the probability that they are unable to do so is known as the "*Calvo probability*".[13]

[11] Joseph S Kong, Nima Sarshar, and Vwani P Roychowdhury. "Experience versus talent shapes the structure of the Web". *Proceedings of the National Academy of Sciences* 105.37 (2008), pp. 13724–13729.

[12] David S. Evans. "The Relationship Between Firm Growth, Size, and Age: Estimates for 100 Manufacturing Industries". *The Journal of Industrial Economics* 35.4 (1987), pp. 567–581.

[13] Guillermo A Calvo. "Staggered prices in a utility-maximizing framework". *Journal of Monetary Economics* 12.3 (1983), pp. 383–398.

Case: $\gamma = 0$ (without rewiring)

- **Proposition:** Assume that $\gamma = 0$ (without rewiring), let τ be the asymptotic fraction of firms with in-degree zero and κ denote the average shifted inverse out-degree.

  (i) Assume that $\rho > 0$. Then the expected number of firms is given by

  $$n_t = \frac{\left(1 - e^{-t\rho\tau}\right)(1 - r\kappa - \delta)}{\rho\tau}, \tag{25}$$

  with the limit

  $$\lim_{t\to\infty} n_t = \frac{1 - \delta - r\kappa}{\rho\tau}.$$

  (ii) Let $a \equiv \frac{\rho\tau}{1-\delta-r\kappa}$, $b \equiv a\,(\delta + r\tilde{\kappa})$, then under the mean field approximation the distribution of the firm's lifetime, T, is given by

  $$\mathbb{P}(T > t) = e^{-(br + a\delta + \rho)t}. \tag{26}$$

(iii) Under the assumption of weak degree correlations, the asymptotic out-degree distribution is given by

$$P^+(k) \propto \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}(2+r(1-\kappa))\Gamma(C^+)\Gamma(C^-)}{\Gamma(1+C^++k)\Gamma(1+C^-+k)}, \tag{27}$$

where

$$C^{\pm} \equiv \frac{2+\delta \pm \sqrt{(2-\delta-r\kappa)^2 - 2r(2\delta+\kappa(2-\delta-r\kappa)) + (\kappa-4)\kappa r^2}}{2(\delta+\kappa r)}.$$

with

$$P^+(k) \sim \left(\frac{1}{\delta+\kappa r}\right)^{k+1}\left(\frac{e}{k}\right)^k k^{\frac{1}{2}-C^+-C^-}$$

for large k, while κ is determined by $\kappa = \sum_{k=0}^{\infty} \frac{1}{1+k} P^+(k)$.

Proof of part (i)

- We assume that $\gamma = 0$ (without rewiring).
- Let N_t be the number of firms at time t. Then we have that

$$N_{t+1} = N_t + 1 - \sum_{i=1}^{N_t} \mathbb{1}_{\{\text{firm } i \text{ exits due to a large shock}\}}$$
$$- \sum_{i=1}^{N_t} \mathbb{1}_{\{\text{firm } i \text{ exits due to a small shock}\}}$$
$$- \sum_{i=1}^{N_t} \mathbb{1}_{\{\text{firm } i \text{ exits because } d_{it}^- = 0\}} \mathbb{1}_{\{d_{it}^- = 0\}}.$$

- Taking the expectation conditional on the filtration $\mathcal{F}_t$ (everything that has happened up to time t) yields

$$\begin{aligned}\mathbb{E}\left(N_{t+1}|\mathcal{F}_t\right) - N_t = 1 &- \sum_{i=1}^{N_t} \mathbb{P}\left(\{\text{firm } i \text{ exits due to a large shock}\}\right) \\ &- \sum_{i=1}^{N_t} \mathbb{P}\left(\{\text{firm } i \text{ exits due to a small shock}\}\right) \\ &- \sum_{i=1}^{N_t} \mathbb{P}\left(\{\text{firm } i \text{ exits because } d_{it}^- = 0\}\right) \mathbb{1}_{\{d_{it}^- = 0\}}.\end{aligned}$$

- Using the fact that

$$\begin{aligned}\mathbb{P}\left(\{\text{firm } i \text{ exits due to a large shock}\}\right) &= \frac{\delta}{N_t}, \\ \mathbb{P}\left(\{\text{firm } i \text{ exits due to a small shock}\}\right) &= \frac{r}{N_t}\frac{1}{1+d_{it}^+} = \frac{r}{N_t}\frac{1}{\tilde{d}_{it}^+}, \\ \mathbb{P}\left(\{\text{firm } i \text{ exits because } d_{it}^- = 0\}\right) &= \rho,\end{aligned}$$

where we have introduced the shifted out-degree $\tilde{d}_{it}^+ \equiv d_{it}^+ + 1$,

- The previous equation can be written as

$$\mathbb{E}\left(N_{t+1}|\mathcal{F}_t\right) - N_t = 1 - \delta - \frac{r}{N_t}\sum_{i=1}^{N_t}\frac{1}{\tilde{d}_{it}^{+}} - \rho N_t \frac{1}{N_t}\sum_{i=1}^{N_t}\mathbb{1}_{\{d_{it}^{-}=0\}}$$
$$= 1 - \delta - r\kappa - \rho\tau N_t, \tag{28}$$

- where

$$\tau = \frac{1}{N_t}\sum_{i=1}^{N_t}\mathbb{1}_{\{d_{it}^{-}=0\}}$$

denotes the fraction of firms with in-degree zero, and

- we have denoted by

$$\kappa \equiv \frac{1}{N_t}\sum_{i=1}^{N_t}\frac{1}{\tilde{d}_{it}^{+}}. \tag{29}$$

- Note that τ converges to $\mathbb{E}(\mathbb{1}_{\{d_{it}^{-}=0\}}) = \mathbb{P}(\{d_{it}^{-}=0\}) = 1 - \mathbb{P}(\{d_{it}^{-}=1\}) = 1 - \mathbb{E}(d_{it}^{-}) = 1 - k_{it}^{-}$.

- Taking the (unconditional) expectation on both sides of Equation (28), and denoting by $n_t = \mathbb{E}\left(N_{t+1}\right) = \mathbb{E}\left(\mathbb{E}\left(N_{t+1}|\mathcal{F}_t\right)\right)$, gives[14]

$$\frac{dn_t}{dt} = 1 - \delta - r\kappa - \rho\tau n_t. \tag{30}$$

- The solution to this differential equation is given by

$$n_t = \left(\rho\tau n_0 + (1 - \delta - r\kappa)(e^{\rho\tau t} - 1)\right) \frac{e^{-\rho\tau t}}{\rho\tau}, \tag{31}$$

- with the limit

$$\lim_{t\to\infty} n_t = \frac{1 - \delta - r\kappa}{\rho\tau}. \tag{32}$$

- With the initial condition, $n_0 = 0$, we get

$$n_t = \frac{\left(1 - e^{-t\rho\tau}\right)(1 - r\kappa - \delta)}{\rho\tau}.$$

[14] T.G. Kurtz. "Limit theorems for sequences of jump Markov processes approximating ordinary differential processes". *Journal of Applied Probability* 8.2 (1971), pp. 344–356.

Case: $\gamma > 0$ (with rewiring)

- **Proposition:** Assume that $\gamma > 0$ (with rewiring), let τ be the asymptotic fraction of firms with in-degree zero and κ denote the average shifted inverse out-degree.

  (i) Assume that $\rho > 0$. Then the expected number of firms is given by

  $$n_t = \frac{\left(1 - e^{-t\rho\tau}\right)(1 - r\kappa - \delta)}{\rho\tau}, \tag{33}$$

  with the limit

  $$\lim_{t\to\infty} n_t = \frac{1 - \delta - r\kappa}{\rho\tau}.$$

  (ii) Denote by $a \equiv \frac{\rho\tau}{1-\delta-r\kappa}\left(1 + \gamma(1 + \bar{k})\right)$, $b \equiv \frac{\rho\tau}{1-\delta-r\kappa}(\delta + r\tilde{\kappa})$, $c \equiv \frac{\rho\tau}{1-\delta-r\kappa}$. Then under the mean field approximation the firm's lifetime, T, is asymptotically distributed as

  $$\mathbb{P}(T > t) \sim e^{-(br+c\delta+\rho)t}$$

  when t becomes large.

(iii) Under the assumption of weak degree correlations, the asymptotic out-degree distribution is given by

$$P^+(k) \propto \frac{1}{(\delta + r\kappa)\Gamma(1 + C^+ + k)\Gamma(1 + C^- + k)} \times \Gamma(k+2)\left(\frac{1 + \gamma\bar{k}(1 - r\kappa + r\tilde{\kappa})}{\delta + \kappa r}\right)^{k+1} \times (2 - r\kappa + r + \gamma\bar{k}(1 - r\kappa + r\tilde{\kappa}))\Gamma(C^+)\Gamma(C^-), \tag{34}$$

with constants $C^{\pm}$ and

$$P^+(k) \sim \left(\frac{1}{\delta + \kappa r}\right)^{k+1} \left(\frac{e}{k}\right)^k k^{\frac{1}{2} - C^+ - C^-},$$

for large k.[15]

[15] In Equation (34) we have that κ is determined by $\kappa = \sum_{k=0}^{\infty} \frac{1}{1+k} P^+(k)$, $\tilde{\kappa}$ is determined by $\tilde{\kappa} = \frac{1}{\bar{k}} \sum_{k=0}^{\infty} \frac{k}{1+k} P^+(k)$, and $\bar{k}$ is determined by $\bar{k} = \sum_{k=0}^{\infty} kP^+(k)$.

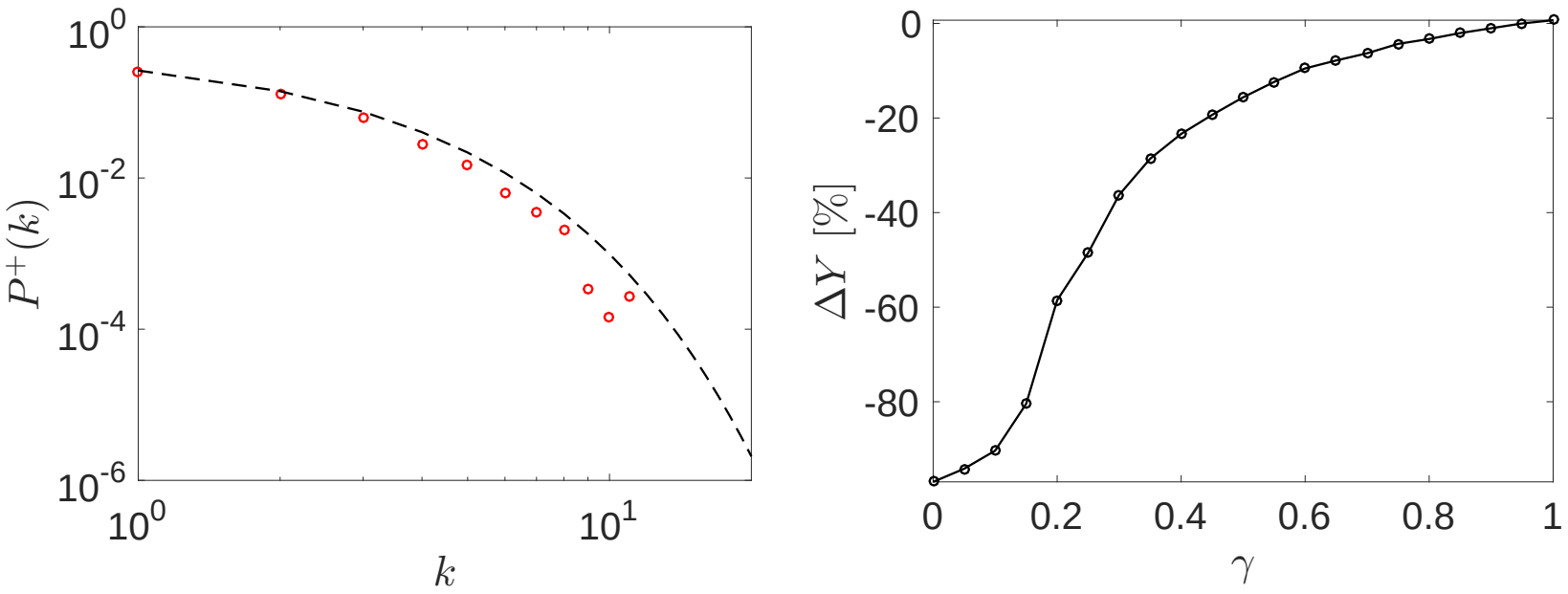

Figure: (Left panel) The out-degree distribution, $P^+(k)$, from a simulation of the stochastic process and the predictions of Equation (34). (Right panel) The relative output loss ΔY (respectively, the number of surviving firms) compared to the case with $\gamma = 1$ (no shock propagation) for varying values of $\gamma \in [0, 1]$.

Adaptivity, Network Concentration and Aggregate Volatility[16]

- Concentration is measured by the out-degree coefficient of variation, CV_k^+,
- Aggregate volatility, σ_n, is defined as the standard deviation of the number of firms n_t over a predefined time window w given by

$$\sigma_n = \sqrt{\sum_{s=t-w}^{t} \left(g_s - \frac{1}{w} \sum_{s'=t-w}^{t} g_{s'} \right)^2},$$

- where the growth rate in the number of firms is given by

$$g_t = \frac{n_t - n_{t-1}}{n_{t-1}}.$$

[16] D. Acemoglu et al. "The network origins of aggregate fluctuations". *Econometrica* 80.5 (2012), pp. 1977–2016.

Higher Adaptivity ⇒ Higher Volatility

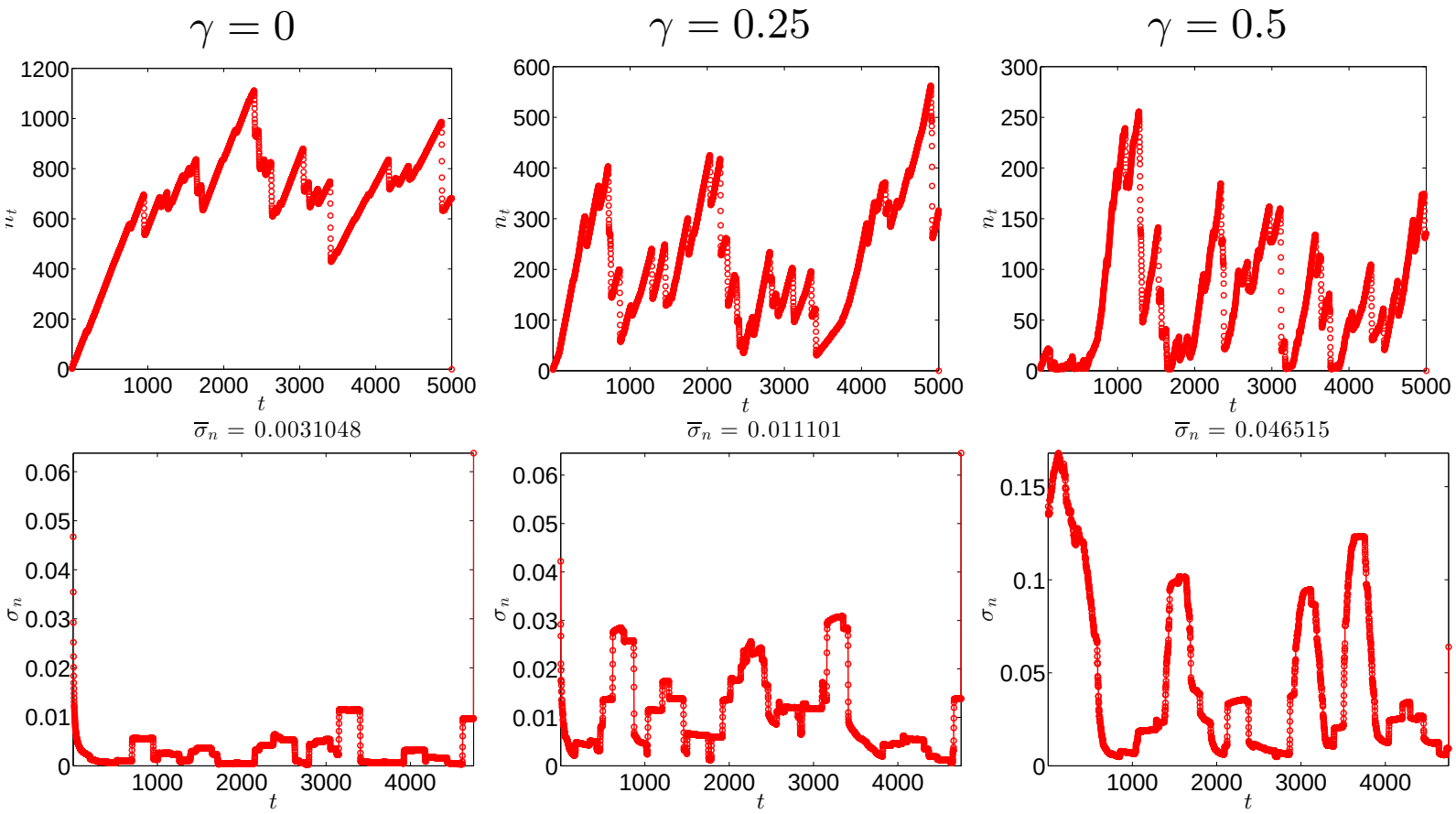

Figure: Assuming that the small shocks are negligible, i.e. we have set $r = 0$, and there is no advantage of large firms to escape the propagation of shocks, we observe that with increasing γ the average number of surviving firms is reduced, while the volatility and the coefficient of variation in the out-degree are increased.

Higher Adaptivity ⇒ Lower Volatility

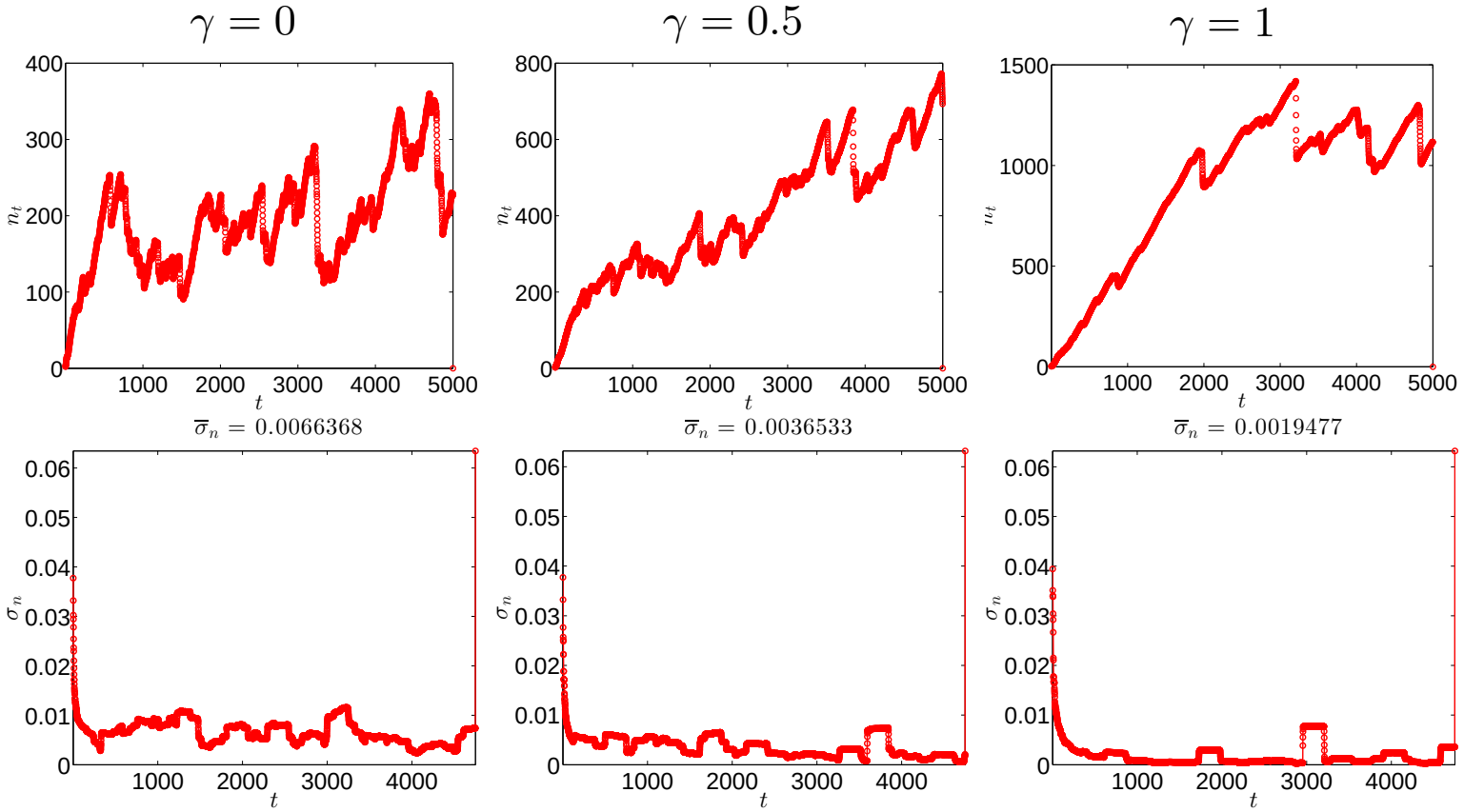

Figure: Assuming non-negligible small shocks, $r = 0.15$, which can be better absorbed by large firms, and preferential rewiring of large firms due to the propagation of shocks, we observe that with increasing γ the average number of surviving firms is increased, while the volatility declines and the coefficient of variation in the out-degree is increasing.

Higher Concentration ⇒ Higher Volatility

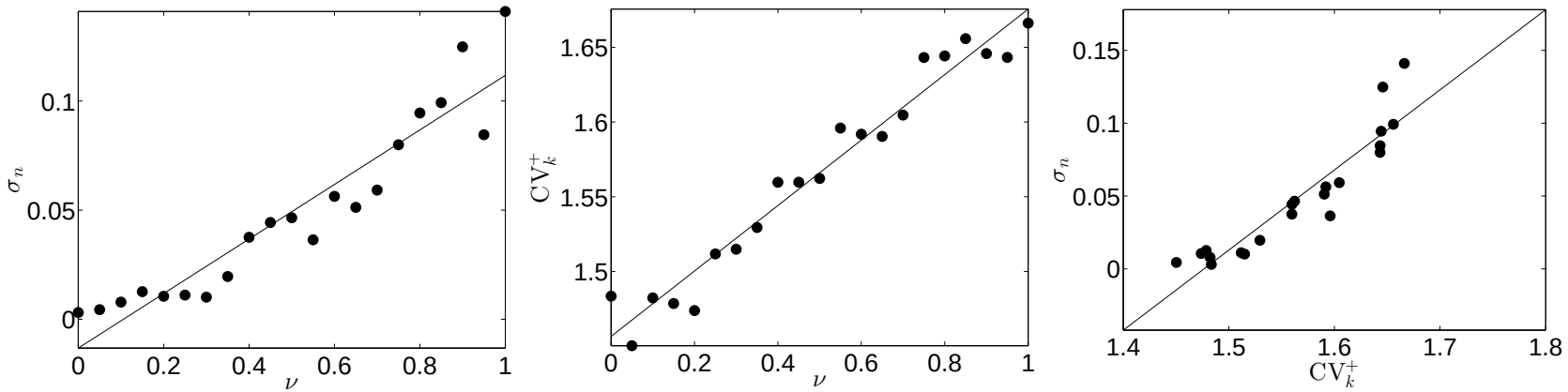

Figure: The volatility, σ_n (left panel), the out-degree coefficient of variation, CV_k^+ (middle panel) and the volatility versus the out-degree coefficient of variation across different values of the rewiring probability γ. Both, σ_n and CV_k^+ are increasing with γ (adaptivity).

Higher Concentration ⇒ Lower Volatility

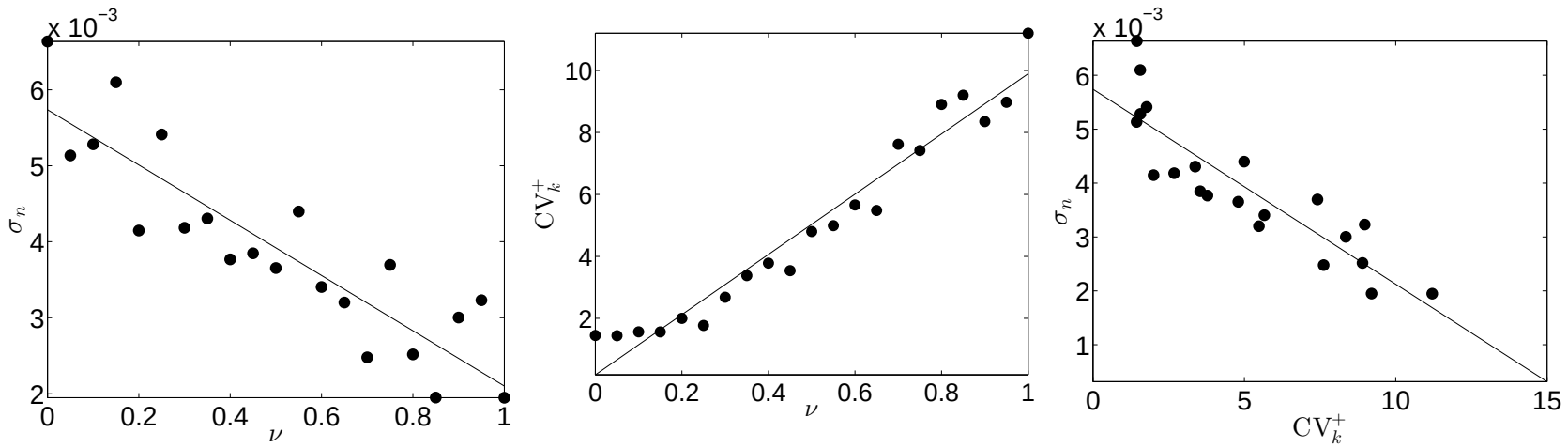

Figure: The volatility, σ_n (left panel), the out-degree coefficient of variation, CV_k^+ (middle panel) and the volatility versus the out-degree coefficient of variation across different values of the rewiring probability ν. While σ_n is decreasing with γ, CV_k^+ is increasing with γ (adaptivity).

Appendix: Proof of part (ii)

- Let d_{it}^{-} be the in-degree of firm i at time t, conditional on not having exited before time t. Then we have that

$$
\begin{aligned}
d_{i,t+1}^{-} = d_{it}^{-} &- \sum_{j=1}^{N_t} a_{ji,t} \mathbb{1}_{\{j \text{ exits because of a large shock}\}} \\
&- \sum_{j=1}^{N_t} a_{ji,t} \mathbb{1}_{\{j \text{ exits because of a small shock}\}} \\
&- \sum_{j=1}^{N_t} a_{ji,t} \mathbb{1}_{\{j \text{ exits because } d_{jt}^{-} = 0\}} \mathbb{1}_{\{d_{jt}^{-}=0\}},
\end{aligned}
$$

for all $t > i$ and the initial condition (at the date of entry) $d_{ii}^{-} = 1$.

- Taking the expectation conditional on $\mathcal{F}_t$ gives

$$\begin{aligned}\mathbb{E}\left(d^-_{i,t+1}|\mathcal{F}_t\right) - d^-_{it} = &-\sum_{j=1}^{N_t} a_{ji,t}\mathbb{P}\left(\{j \text{ exits because of a large shock}\}\right)\\ &-\sum_{j=1}^{N_t} a_{ji,t}\mathbb{P}\left(\{j \text{ exits because of a small shock}\}\right)\\ &-\sum_{j=1}^{N_t} a_{ji,t}\mathbb{P}\left(\{j \text{ exits because } d^-_{jt}=0\}\right)\mathbb{1}_{\{d^-_{jt}=0\}}.\end{aligned}$$

- Using the fact that $\mathbb{P}\left(\pi_{it} - \zeta < 0\right) = \frac{1}{\tilde{d}^+_{it}}$, and that

$$\begin{aligned}\mathbb{P}\left(\{\text{firm } i \text{ exits due to a large shock}\}\right) &= \frac{\delta}{N_t},\\ \mathbb{P}\left(\{\text{firm } i \text{ exits due to a small shock}\}\right) &= \frac{r}{N_t}\frac{1}{\tilde{d}^+_{it}},\\ \mathbb{P}\left(\{\text{firm } i \text{ exits because } d^-_{it}=0\}\right) &= \rho,\end{aligned}$$

- we then get

$$\mathbb{E}\left(d^-_{i,t+1}|\mathcal{F}_t\right) - d^-_{it} = -\sum_{j=1}^{N_t} a_{ji,t}\frac{\delta}{N_t} - \sum_{j=1}^{N_t} a_{ji,t}\frac{r}{N_t}\frac{1}{\tilde{d}^+_{jt}} - \sum_{j=1}^{N_t} a_{ji,t}\rho\tau.$$

- We can write this as follows

$$\mathbb{E}\left(d^-_{i,t+1}|\mathcal{F}_t\right) - d^-_{it} = -\frac{1}{N_t}\sum_{j=1}^{N_t} a_{ji,t}\left(\delta + \frac{r}{\tilde{d}^+_{jt}} + \rho\tau N_t\right). \tag{35}$$

- Taking the (unconditional) expectation on both sides of Equation (35), denoting by $k^-_{i,t+1} = \mathbb{E}\left(d^-_{i,t+1}\right) = \mathbb{E}\left(\mathbb{E}\left(d^-_{i,t+1}|\mathcal{F}_t\right)\right)$ and $\bar{k}_t \equiv \frac{1}{N_t}\sum_{j=1}^{N_t}\tilde{d}^+_{jt}$, gives[17]

$$\frac{dk^-_{it}}{dt} = -\frac{1}{n_t}k^-_{it}\left(\delta + \frac{r}{1+\bar{k}_t} + \rho\tau n_t\right).$$

[17] T.G. Kurtz. "Limit theorems for sequences of jump Markov processes approximating ordinary differential processes". *Journal of Applied Probability* 8.2 (1971), pp. 344–356.

- Inserting the asymptotic number of firms from Equation (32) gives

$$\begin{aligned}\frac{dk_{it}^-}{dt} &= -\frac{\rho\tau}{1-\delta-r\kappa}k_{it}^-\left(\delta+\frac{r}{1+\bar{k}_t}+\rho\tau\frac{1-\delta-r\kappa}{\rho\tau}\right)\\ &= -\frac{\rho\tau}{1-\delta-r\kappa}k_{it}^-\left(1+r\left(\frac{1}{1+\bar{k}_t}-\kappa\right)\right)\\ &= -\frac{\rho\tau}{1-\delta-r\kappa}k_{it}^-\left(1+r\left(\frac{1}{1+\bar{k}_t}-\kappa\right)\right).\end{aligned}$$

- When $(1+\bar{k}_t)^{-1} \approx \kappa = \lim_{t\to\infty}\frac{1}{N_t}\sum_{i=1}^{N_t}\frac{1}{1+d_{it}^+}$ this can be written as

$$\frac{dk_{it}^-}{dt} = -\frac{\rho\tau}{1-\delta-r\kappa}k_{it}^-, \tag{36}$$

- with the initial condition $k_{ii}^- = 1$, and the solution

$$k_{it}^- = e^{-\frac{(t-i)\rho\tau}{1-\delta-r\kappa}}. \tag{37}$$

- This shows that the in-degree of firm i is exponentially decaying.

- Next, let $\tilde{d}_{it}^{+}$ be the shifted out-degree of firm i at time t, conditional on not having exited before time t.
- Then we have that

$$\tilde{d}_{i,t+1}^{+} = \tilde{d}_{it}^{+} + \mathbb{1}_{\{i \to t\}} - \sum_{j=1}^{N_t} a_{ij,t} \mathbb{1}_{\{j \text{ exits due to a large shock}\}}$$
$$- \sum_{j=1}^{n_t} a_{ij,t} \mathbb{1}_{\{j \text{ exits due to a small shock}\}},$$

for all $t > i + 1$ and the initial condition $\tilde{d}_{ii}^{+} = 1$.

- Taking the expectation conditional on $\mathcal{F}_t$ gives

$$\mathbb{E}\left(\tilde{d}_{i,t+1}^{+} | \mathcal{F}_t\right) - \tilde{d}_{it}^{+} = \mathbb{P}\left(\{i \to t\}\right) - \sum_{j=1}^{N_t} a_{ij,t} \mathbb{P}\left(\{j \text{ exits due to a large shock}\}\right)$$
$$- \sum_{j=1}^{N_t} a_{ij,t} \mathbb{P}\left(\{j \text{ exits due to a small shock}\}\right).$$

- Using the fact that

$$\mathbb{P}\left(\{i \to t\}\right) = \frac{1}{N_t},$$

$$\mathbb{P}\left(\{\text{firm } j \text{ exits due to a large shock}\}\right) = \frac{\delta}{N_t},$$

$$\mathbb{P}\left(\{\text{firm } j \text{ exits due to a small shock}\}\right) = \frac{r}{N_t}\frac{1}{\tilde{d}^{+}_{jt}},$$

- we can write this as follows

$$\mathbb{E}\left(\tilde{d}^{+}_{i,t+1}|\mathcal{F}_t\right) - \tilde{d}^{+}_{it} = \frac{1}{N_t} - \sum_{j=1}^{N_t} a_{ij,t}\frac{\delta}{N_t} - \sum_{j=1}^{N_t} a_{ij,t}\frac{r}{N_t}\frac{1}{\tilde{d}^{+}_{jt}}.$$

- This can be written as

$$\mathbb{E}\left(\tilde{d}^{+}_{i,t+1}|\mathcal{F}_t\right) - \tilde{d}^{+}_{it} = \frac{1}{N_t} - \sum_{j=1}^{N_t} a_{ij,t}\left(\frac{\delta}{N_t} + \frac{r}{N_t}\frac{1}{\tilde{d}^{+}_{jt}}\right).$$

- Taking the expectation on both sides of the equation and denoting by $k^{+}_{it} = \mathbb{E}\left(\tilde{d}^{+}_{it}\right)$ yields[18]

$$\frac{dk^{+}_{it}}{dt} = \frac{1}{n_t} - \left(\delta + \frac{r}{1+\bar{k}_t}\right)\frac{k^{+}_{it}}{n_t}.$$

[18] T.G. Kurtz. "Limit theorems for sequences of jump Markov processes approximating ordinary differential processes". *Journal of Applied Probability* 8.2 (1971), pp. 344–356.

- In the stationary state we have that $\bar{k}_t = \bar{k}$, $\lim_{t\to\infty} n_t = \frac{1-\delta-r\kappa}{\rho\tau}$, so that we can write this as follows

$$\frac{dk_{it}^{+}}{dt} = a - bk_{it}^{+},$$

- where we have denoted by

$$a \equiv \frac{\rho\tau}{1-\delta-r\kappa},$$
$$b \equiv \frac{\rho\tau}{1-\delta-r\kappa}\left(\delta + r\tilde{\kappa}\right) = a\left(\delta + r\tilde{\kappa}\right).$$

- With the initial condition $k_{ii}^{+} = 1$ the solution is given by

$$k_{it}^{+} = \frac{a + (b-a)e^{-b(t-i)}}{b}, \tag{38}$$

- With the limit
$$\lim_{t\to\infty} k_{it}^{+} = \frac{a}{b} = (\delta + r\tilde{\kappa})^{-1}.$$
- Next, let S_{it} be the indicator variable indicating whether firm i is still alive at time t. Then we have that
$$\begin{aligned} S_{i,t+1} = S_{it} \Big(1 &- \mathbb{1}_{\{i \text{ exits due to a large shock}\}} \\ &- \mathbb{1}_{\{i \text{ exits due to a small shock}\}} \\ &- \mathbb{1}_{\{i \text{ exits because } d_{it}^{-} = 0\}} \mathbb{1}_{\{d_{it}^{-}=0\}} \Big). \end{aligned}$$
- Taking the expectation with respect to $\mathcal{F}_t$ gives
$$\begin{aligned} \mathbb{E}\left(S_{i,t+1} | \mathcal{F}_t\right) = S_{it} \Big(1 &- \mathbb{P}\left(\{i \text{ exits due to a large shock}\}\right) \\ &- \mathbb{P}\left(\{i \text{ exits due to a small shock}\}\right) \\ &- \mathbb{P}\left(\{i \text{ exits because } d_{it}^{-} = 0\}\right) \mathbb{1}_{\{d_{it}^{-}=0\}} \Big). \end{aligned}$$

- Using the fact that

$$\mathbb{P}\left(\{\text{firm } i \text{ exits due to a large shock}\}\right) = \frac{\delta}{N_t},$$
$$\mathbb{P}\left(\{\text{firm } i \text{ exits due to a small shock}\}\right) = \frac{r}{N_t}\frac{1}{\tilde{d}_{it}^{+}},$$
$$\mathbb{P}\left(\{\text{firm } i \text{ exits because } d_{it}^{-} = 0\}\right) = \rho,$$

gives

$$\mathbb{E}\left(S_{i,t+1}|\mathcal{F}_t\right) = S_{it}\left(1 - \frac{\delta}{N_t} - \frac{r}{N_t}\frac{1}{\tilde{d}_{it}^{+}} - \rho\mathbb{1}_{\{d_{it}^{-}=0\}}\right),$$

- or equivalently

$$\mathbb{E}\left(S_{i,t+1}|\mathcal{F}_t\right) - S_{it} = -S_{it}\left(\frac{\delta}{N_t} + \frac{r}{N_t}\frac{1}{\tilde{d}_{it}^{+}} + \rho\mathbb{1}_{\{d_{it}^{-}=0\}}\right).$$

- Taking the expectation and denoting by $s_{it} = \mathbb{E}\left(S_{it}\right)$ and $k_{it}^{+} = \mathbb{E}\left(\tilde{d}_{it}^{+}\right)$ then gives[19]

$$\frac{ds_{it}}{dt} = -s_{it}\left(\frac{\delta}{n_t} + \frac{r}{n_t}\frac{1}{k_{it}^{+}} + \rho\mathbb{P}\left(\{d_{it}^{-} = 0\}\right)\right), \tag{39}$$

  with the initial condition $s_{ii} = 1$.
- Note that by assumption d_{it}^{-} can be either zero or one, so that

$$\begin{aligned}\mathbb{P}\left(\{d_{it}^{-} = 0\}\right) &= 1 - \mathbb{P}\left(\{d_{it}^{-} = 1\}\right)\\ &= 1 - \mathbb{E}\left(d_{it}^{-}\right)\\ &= 1 - k_{it}^{-}.\end{aligned} \tag{40}$$

[19] T.G. Kurtz. "Limit theorems for sequences of jump Markov processes approximating ordinary differential processes". *Journal of Applied Probability* 8.2 (1971), pp. 344–356.

- Inserting Equation (40) into Equation (39) gives

$$\frac{ds_{it}}{dt} = -s_{it}\left(\frac{\delta}{n_t} + \frac{r}{n_t}\frac{1}{k_{it}^{+}} + \rho(1 - k_{it}^{-})\right), \tag{41}$$

- With k_{it}^{-} from Equation (37) and k_{it}^{+} from Equation (38) we then get

$$\frac{ds_{it}}{dt} = -s_{it}\left(a\delta + \frac{arb}{a + (b-a)e^{-b(t-i)}} + \rho\left(1 - e^{-a(t-i)}\right)\right),$$

- or equivalently

$$\frac{d\ln s_{it}}{dt} = -\left(a\delta + \frac{arb}{a + (b-a)e^{-b(t-i)}} + \rho\left(1 - e^{-a(t-i)}\right)\right). \tag{42}$$

- With the initial condition $s_{ii} = 1$ the solution is given by

$$
\begin{aligned}
s_{it} = & \left(\frac{b}{b - a\left(1 - e^{b(t-i)}\right)} \right)^{r} \\
& \times \exp\left(\frac{\rho}{b} \left(e^{-a(t-i)} \left(\frac{a e^{b(t-i)}}{a-b} \right)^{a/b} \left(B_{\frac{a e^{b(t-i)}}{a-b}} \left(-\frac{a}{b}, 0 \right) \right.\right.\right. \\
& \left. - B_{\frac{a e^{b(t-i)}}{a-b}} \left(1 - \frac{a}{b}, 0 \right) \right) \\
& \left.\left. + \left(\frac{a}{a-b} \right)^{a/b} \left(B_{\frac{a}{a-b}} \left(1 - \frac{a}{b}, 0 \right) - B_{\frac{a}{a-b}} \left(-\frac{a}{b}, 0 \right) \right) \right) - (t-i)(a\delta + \rho) \right).
\end{aligned}
$$

- When $a + (b-a)e^{-b(t-i)} \approx a$ and $1 - e^{-a(t-i)} \approx 1$ for large t we can simplify Equation (42) to

$$
\frac{d \ln s_{it}}{dt} = -\left(a\delta + rb + \rho\right),
$$

- and the solution is given by

$$
s_{it} = e^{-(br + a\delta + \rho)(t-i)}, \tag{43}
$$

which is an exponentially decaying function.

Appendix: Proof of part (iii)

- Let $N_t^+(k)$ denote the number of firms with out-degree k at time t.
- Taking the expectation with respect to $\mathcal{F}_t$ gives

$$\begin{aligned}\mathbb{E}\left(N_{t+1}^+(k)|\mathcal{F}_t\right) - N_t^+(k) = &\frac{1}{N_t}N_t^+(k-1) - \frac{1}{N_t}N_t^+(k) - \frac{\delta}{N_t}N_t^+(k) \\ &- \frac{r}{N_t}\frac{1}{1+k}N_t^+(k) - \rho N_t^{-+}(0,k) \\ &- kN_t^+(k)\left(\frac{\delta}{N_t} + \frac{r}{N_t}\sum_{l=0}^{\infty}P_t^+(l|k)\frac{1}{l+1},\right),\end{aligned}$$

- where $N_t^{-+}(0,k)$ denotes the number of firms with in-degree 0 and out-degree k and $P_t^+(l|k) \equiv \mathbb{P}(d_{j,t}^+ = l|d_{i,t} = k, a_{ij,t} = 1)$.

- Assuming that there are only weak degree correlations we then can write

$$
\begin{aligned}
N_t^{-+}(0,k) &= \mathbb{P}\left(d_{it}^{+}=k, d_{it}^{-}=0\right) N_t \\
&= \left(\mathbb{P}\left(d_{it}^{+}=k\right)\mathbb{P}\left(d_{it}^{-}=0\right) + \mathrm{Cov}\left(\mathbb{1}_{\{d_{it}^{-}=0\}}, \mathbb{1}_{\{d_{it}^{+}=k\}}\right)\right) N_t \\
&\approx \mathbb{P}\left(d_{it}^{+}=k\right)\mathbb{P}\left(d_{it}^{-}=0\right) N_t \\
&= \frac{N_t^{+}(k) N_t^{-}(0)}{N_t},
\end{aligned}
$$

- where we have assumed that $\mathrm{Cov}\left(\mathbb{1}_{\{d_{it}^{-}=0\}}, \mathbb{1}_{\{d_{it}^{+}=k\}}\right) \approx 0$, and

$$
P_t^{+}(l|k) \equiv \mathbb{P}(d_{j,t}^{+}=l|d_{i,t}=k, a_{ij,t}=1) \approx \mathbb{P}(d_{j,t}^{+}=l) = \frac{N_t^{+}(l)}{N_t}.
$$

- Inserting gives

$$
\begin{aligned}
&\mathbb{E}\left(N_{t+1}^{+}(k)|\mathcal{F}_t\right) - N_t^{+}(k) \\
&= \frac{1}{N_t} N_t^{+}(k-1) - \frac{1}{N_t} N_t^{+}(k) - \frac{\delta}{N_t} N_t^{+}(k) - \frac{r}{N_t} \frac{1}{1+k} N_t^{+}(k) - \rho\tau N_t^{+}(k) \\
&- k N_t^{+}(k) \left(\frac{\delta}{N_t} + \frac{r}{N_t} \sum_{l=0}^{\infty} \frac{N_t^{+}(l)}{N_t} \frac{1}{l+1} \right) \\
&= \frac{1}{N_t} N_t^{+}(k-1) - \frac{1}{N_t} N_t^{+}(k) - \frac{\delta}{N_t} N_t^{+}(k) - \frac{r}{N_t} \frac{1}{1+k} N_t^{+}(k) - \rho\tau N_t^{+}(k) \\
&- k \left(\frac{\delta}{N_t} + \frac{r}{N_t} \kappa \right) N_t^{+}(k),
\end{aligned}
$$

where $\tau = \frac{N_t^{-}(0)}{N_t}$ denotes the asymptotic fraction of firms with in-degree zero and $\kappa \equiv \sum_{k=0}^{\infty} \frac{N_t^{+}(k)}{N_t} \frac{1}{k+1} = \sum_{k=0}^{\infty} P_t^{+}(k) \frac{1}{k+1}$.

- We then can write[20]

$$\frac{dN_t^+(k)}{dt} = \frac{1}{N_t}\left(N_t^+(k-1) - \left(1+\delta+\frac{r}{1+k}+\rho N_t^-(0)+k(\delta+r\kappa)\right)N_t^+(k)\right).$$

- In the stationary state, where $\frac{dN_t^+(k)}{dt} = 0$, we then have that

$$N^+(k) = \frac{1}{1+\delta+\frac{r}{1+k}+\rho N^-(0)+k(\delta+r\kappa)}N^+(k-1),$$

[20] T.G. Kurtz. "Limit theorems for sequences of jump Markov processes approximating ordinary differential processes". *Journal of Applied Probability* 8.2 (1971), pp. 344–356.

- From the above we get

$$N^+(k) = N^+(0) \prod_{l=1}^{k} \frac{1}{1+\delta+\frac{r}{1+l}+\rho N^-(0)+(\delta+r\kappa)l}$$
$$= N^+(0) \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}(1+\delta+N^-(0)\rho+r)\Gamma(C^+)\Gamma(C^-)}{\Gamma(1+C^++k)\Gamma(1+C^-+k)},$$

- where

$$C^\pm \equiv \frac{1}{2(\delta+\kappa r)} \times \Big(1+\kappa r+2\delta+N^-(0)\rho$$
$$\pm\sqrt{(N^-(0)\rho+1)^2-2r(2\delta+\kappa+\kappa N^-(0)\rho)+(\kappa-4)\kappa r^2}\Big).$$

- Note that $N^+(0) = N - \sum_{k=1}^{\infty} N^+(k)$, and consequently

$$N^+(0) = N\Bigg(1+(1+\delta+N^-(0)\rho+r)\Gamma(C^+)\Gamma(C^-)$$
$$\times \sum_{k=1}^{\infty} \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1+C^++k)\Gamma(1+C^-+k)}\Bigg)^{-1}.$$

- The out-degree distribution $P^+(k) = \frac{N^+(k)}{N}$ is then given by

$$P^+(k) = \Bigg(1 + (1 + \delta + N^-(0)\rho + r)\Gamma(C^+)\Gamma(C^-) \times \sum_{k=1}^{\infty} \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1+C^+ + k)\Gamma(1+C^- + k)}\Bigg)^{-1} \times \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}(1 + \delta + N^-(0)\rho + r)\Gamma(C^+)\Gamma(C^-)}{\Gamma(1+C^+ + k)\Gamma(1+C^- + k)}.$$

- For large k we have that

$$\begin{aligned}\frac{\Gamma(k+2)}{\Gamma(C^+ + k + 1)\Gamma(C^- + k + 1)} &\sim \frac{k\Gamma(k+1)}{k^{C^+}k^{C^-}\Gamma(k+1)\Gamma(k+1)} \\ &\sim \frac{k^{-C^+ - C^- + 1}}{\Gamma(k+1)} \\ &\sim \frac{e^k k^{-C^+ - C^- - k + \frac{1}{2}}}{\sqrt{2\pi}},\end{aligned}$$

- where we have used the fact that for large k, $\lim_{k\to\infty} \frac{\Gamma(k+\alpha)}{\Gamma(k)k^{\alpha}} = 1, \alpha \in \mathbb{R}$, and Stirling's formula, $\Gamma(k+1) \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$, as $k \to \infty$.
- The out-degree distribution is then asymptotically given by

$$P^+(k) \sim \left(\frac{1}{\delta + \kappa r}\right)^{k+1} \left(\frac{e}{k}\right)^k k^{\frac{1}{2} - C^+ - C^-}.$$

- Next, recall that the number of firms, N_t, evolves according to

$$\frac{dN_t}{dt} = 1 - \delta - r\kappa_t - \rho N_t^-(0),$$

with $\kappa_t = \sum_{k=0}^{\infty} \frac{1}{1+k} P_t^+(k)$,

- so that in the stationary state we obtain

$$N^-(0) = \frac{1 - \delta - r\kappa}{\rho}.$$

- Inserting yields

$$P^+(k) = \left(1 + (2 + r(1-\kappa))\Gamma(C^+)\Gamma(C^-)\sum_{k=1}^{\infty}\frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1+C^+ + k)\Gamma(1+C^- + k)}\right)^{-1}$$
$$\times \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}(2 + r(1-\kappa))\Gamma(C^+)\Gamma(C^-)}{\Gamma(1+C^+ + k)\Gamma(1+C^- + k)},$$

- where

$$C^{\pm} \equiv \frac{2 + \delta \pm \sqrt{(2-\delta-r\kappa)^2 - 2r(2\delta + \kappa(2-\delta-r\kappa)) + (\kappa-4)\kappa r^2}}{2(\delta+\kappa r)}.$$

while κ is determined by

$$\kappa = \sum_{k=0}^{\infty}\frac{1}{1+k}P^+(k).$$

- This nonlinear equation can be solved for κ numerically using for example an iterated fixed point algorithm.