Abstract

This paper investigates the impact of a change in aggregate credit supply on firm dynamics in an economy with financial frictions. We model a lifetime lending relationship between banks and firms in a general equilibrium framework with households making endogenous occupational decision. Financial markets are endogenously incomplete due to asymmetric information, and financial constraints emerge from the optimal financial contract. In equilibrium, we determine the interest rate and wage rate, as well as the share of entrepreneurs in the economy. Following the optimal contract, the distribution of firm size and firm dynamics such as size, growth, and volatility of growth are derived. Further, we introduce reserve requirements on banks, which induce tighter aggregate credit supply. The model shows that, as reserve ratio (i.e., the share of deposits as reserve) increases, the interest rate increases, the wage rate decreases, and the share of entrepreneur decreases in equilibrium. Aggregate capital supply and aggregate output decrease. In addition, not only do firms operate in smaller sizes on average, but also there is a reallocation of credit from young firms towards older ones, which implies a tighter financial constraint on the young firms in the economy.

Keywords: Dynamic contracts, asymmetric information, firm dynamics, general equilibrium

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1 Introduction

After the financial crisis of 2007-2008, one focus of public debate is the role of financial markets in shaping aggregate fluctuations and firm dynamics. Previous studies on financial frictions emphasize their role in amplifying business cycle fluctuations with aggregate shocks from the production sector. This paper primarily focuses on disruptions within the financial sector (i.e., “financial shocks”). We aim at understanding the long-run impact of tightening aggregate credit supply on macroeconomic aggregates, as well as on important aspects of firm dynamics such as firm size, growth rate, and the volatility of growth over a firm’s lifetime. We characterize a long-term lending relationship between banks and firms in a general equilibrium framework with households making endogenous occupational decisions. Financial markets are endogenously incomplete, and financial constraints emerge from optimal financial contracts. Among the results, we highlight the disproportionate effect on the credit availability to young firms. The paper shows that there is a reallocation of credit from younger firms towards older ones.

The key structure of the model is a microfounded long-term lending relationship between financial intermediaries and firms in a frictional financial market. We characterize an infinite time horizon model where firms need credit from banks to finance their production. A key assumption is asymmetric information: Firms are subject to an idiosyncratic shock to productivity each period, either high or low, which is not verifiable by the banks. Therefore, the profit-maximizing banks offer firms a lifetime financial contract that is incentive-compatible and satisfies a limited liability constraint. Following the literature on dynamic contracting under asymmetric information, the optimal financial contract is derived from recursive formulation using entrepreneurs’ promised value (i.e., entrepreneurs’ claims to future cash flows) as a state variable. It consists of the level of bank loans, state-contingent repayments and entrepreneurs’ future continuation utilities as functions of the state variable.\footnote{Literature on dynamic contracting under asymmetric information includes, for example, Green} A key feature of the dynamic contract is that the
state variable summarizes firms’ history of productivity shocks, which is interpreted as a proxy for firm performance. As a result, the financial contract is contingent on firms’ previous performance: Overall, firms that have experienced a sequence of high productivity shocks tend to get more loans and become larger on size (i.e., employ more capital and labor), and vice versa. The dynamic contract reflects two salient features of financial markets: First, long-term lending relationships help to overcome informational asymmetries between creditors and firms. And second, terms of contracts depend on firms’ past performance.

The optimal financial contract determines the distribution of firm size in the economy and firm dynamics – firm size, growth rate and the volatility of growth over a firm’s lifetime. We show that the results are consistent with empirical regularities: Firm distribution in terms of their claim to future cash flows skews to the right when firms are young. As firms become older, such skewness diminishes and the distribution becomes more symmetric. This implies that as a result of financial frictions, the optimal contract imposes an endogenous credit constraint on firms, especially on the young ones. Such constraint loosens gradually as firms grow older. Moreover, there is a positive correlation between firm size and firm age. The growth rate of younger firms is larger on average and more volatile than that of older firms.

Further, we incorporate the dynamic contract in a general equilibrium framework. Workers save in banks and supply labor in firms. Entrepreneurs are financed by banks and run firms with capital and labor as inputs. To characterize households’ occupational


Using data from manufacturing firms in Portugal and Italy, respectively, Cabral and Mata (2003) and Angelini and Generale (2008) show empirically that the distribution of firm size evolves from right-skewed towards symmetric as firms age.

Cooley and Quadrini (2001) and Clementi and Hopenhayn (2006) show such patterns of firm dynamics over lifetime under asymmetric information and under limited enforcement of financial contracts, respectively.
decisions, the newborn households, who are *ex ante* identical, decide endogenously to become workers or entrepreneurs. Therefore, the share of entrepreneurs in the economy is determined in equilibrium. The general equilibrium model consists of the following components: (i) Lifetime-expected-utility-maximizing workers who decide consumption and saving, labor and leisure. (ii) Profit-maximizing banks that choose the optimal dynamic contract (i.e., bank loans, state-contingent repayment, and future promised values as a function of entrepreneurs’ promised value today). (iii) Utility-maximizing entrepreneurs who decide capital and labor employment for a given level of bank loans. The model generates a stationary equilibrium that accommodates firm dynamics (i.e., there are firms of all ages and with all histories of productivity realization in the economy). In particular, we introduce a constant entry of newborn households and an exogenous exit of the existing ones in each period. By combining the three partial decision problem (i)-(iii) with the distribution of workers and of entrepreneurs, we close our model and determine the interest rate, wage rate, and the share of entrepreneurs under the stationary general equilibrium.

With this framework, we investigate the long-run impact of higher aggregate credit supply on the share of entrepreneurs, the factor prices, and the lending policy of the financial intermediaries, as well as on the resulting firm dynamics and on the macroeconomic aggregates in equilibrium. To do this, we introduce a constraint on banks’ balance sheets. Namely, an exogenous share of deposits (defined as the reserve ratio) needs to be saved as reserves. Then we compare the stationary equilibrium under different levels of reserve ratio. We first calibrate our model to the U.S. economy and solve for the general equilibrium numerically. Our model predicts that for the calibrated economy, a rise in the reserve ratio from zero to 20% leads to a 1.68% increase in the interest rate from 0.0417 to 0.0424, and a 1.87% decrease in the share of entrepreneurs from 0.0762 to 0.0757 in the long run. Moreover, such a change in our model depresses the aggregate capital supply by 1.20% and the aggregate output by 0.70%. The negative impact on the aggregate output is mitigated by a decline in labor cost and thus a higher labor intensity, which
increases by 0.72% from 1.520 to 1.531, in equilibrium.

A key result of the paper is the reallocation effect of credit among entrepreneurs of different ages due to a decrease in aggregate credit availability. Specifically, bank credit to firms decreases by 0.068% on average as the reserve ratio increases from zero to 20%. However, the average credit to firms below 20 years old (accounting for 81.13% of all firms) decreases by 0.138%, twice as much as the average decrease. At the same time, the average credit to older firms (age above 20, accounting for 18.87% of all firms) even increases by 0.22%. This implies that there is a reallocation of credit from young firms towards older ones. In other words, a decrease in aggregate credit depresses average firm size due to a tighter financial constraint on the young firms. We show that the reallocation of credit is a result of the equilibrium price effect on the lending policy and on households’ occupational choices.

Our paper incorporates dynamic financial contracts into a general equilibrium framework with an endogenous share of entrepreneurs and firm dynamics. There are a few papers that work in a similar modeling framework. Smith and Wang (2006) investigate the long-term dynamic credit relationship between borrowers and lenders with asymmetric information in general equilibrium. The borrowers produce with exogenously fixed levels of capital and labor inputs. We complement their framework by adopting a more realistic, decreasing-return-to-scale firm production function. This allows us, on the one hand, to characterize the heterogeneity in firm size: Namely, the model delivers implications for firm size distribution and firm dynamics. More importantly, we can investigate the distributional consequences, as a result of tightening aggregate credit supply, on the credit availability among firms of different ages and sizes. On the other hand, the model includes both the supply channel and the demand channel. The latter comes from the optimal investment decisions of profit-maximizing entrepreneurs. Therefore, the equilibrium is

simultaneously determined by the two channels.

Recent studies that are closely related to our paper include Dyrda (2015), Gross and Verani (2013) and Verani (2015). These papers all incorporate dynamic contracts in a general equilibrium framework with financial frictions and firms making optimal investment decisions. Compared with their work, we endogenize households’ occupational decisions, and determine the share of entrepreneurs in equilibrium. The contribution is two-fold: First, from a technical perspective, more complex equilibrium conditions are considered, which raises computational challenges in finding a unique stationary equilibrium in the economy. Second, and more importantly, households’ endogenous occupational choices provide important implications for the macroeconomy as factor prices change. Consider, for instance, that there is a decrease in aggregate credit supply. The capital market clearing condition implies that the interest rate increases. Higher capital costs in firm production decrease firm profit and thus firms’ repayment to banks. Therefore, bank profit decreases. Competitive banks promise lower utility (i.e., lower initial promised value) to newborn firms, which maps into lower bank loans and tighter financial constraints. So far, the mechanism comes from the bank lending channel. In addition to this, our framework further shows that an increase in interest rate at the same time increases workers’ lifetime expected utility. The higher utility attracts newborn households to become workers and the share of entrepreneurs declines. Therefore, in an economy where households can endogenously choose their occupations, the negative effect on entrepreneurs’ lifetime expected utility and thus their credit constraint is mitigated. However, the decline in aggregate output is amplified due to a lower share of entrepreneurs in equilibrium.

In addition, this paper contributes to the broad literature on financial frictions and their macroeconomic importance. Carlstrom and Fuerst (1997), Bernanke et al. (1999) and Dyrda (2015) analyze the impact of fluctuations in microeconomic uncertainty on firm financing. Gross and Verani (2013) study the impact of financial constraints on exporter dynamics. Verani (2015) investigates the impact of financial friction on macroeconomic aggregates in a framework with costly monitoring and limited enforcement. Quadrini (2011) and Brunnermeier et al. (2013) provide a comprehensive review of literature in this
Miao and Wang (2010) investigate the role of financial frictions in transmitting aggregate fluctuations into the real economy. Our paper is distinguished from this strand of literature in two aspects: First, instead of assuming linearity of firm technology and preferences, we incorporate decreasing-return-to-scale production functions and entrepreneurs with concave utility functions. As a result, on the one hand, credit distribution among firms delivers implications for the macroeconomy. To see this more clearly, marginal return to capital differs among firms in equilibrium, and thus allocation of credit among firms influences the aggregate output in the economy. On the other hand, there is risk-sharing between financial intermediaries and firms. Second, these papers study shocks that arise within the production sector (e.g., productivity shock). A common role of financial frictions is thus an amplification effect in generating macroeconomic consequences (e.g., economic recessions). Nevertheless, we focus on the disruption within the financial sector: higher reserve requirements decreasing credit supply of banks. In addition to influencing the macroeconomic aggregates, financial frictions result in a reallocation effect of credit from younger firms to older ones. Therefore, narrowly speaking, we contribute to the literature on financial shocks and firms’ asymmetric responses. This effect is investigated with empirical evidence in many studies. Our paper provides a theoretical framework where, among others, the reallocation effect is an equilibrium outcome, and the underlying mechanism is investigated.

7 Recent papers on this include Dyrda (2015) and Verani (2015), who have decreasing-return-to-scale production functions but study fluctuations in firm productivity realizations.

8 There are fewer papers that study shocks within the financial sector that generate aggregate fluctuations. Recent papers, among others, include Christiano et al. (2010), Gertler and Karadi (2011), and Jermann and Quadrini (2012).

9 For example, Gertler and Gilchrist (1994) show that small firms account for a significantly disproportionate share of the manufacturing decline that follows tightening of monetary policy. Fort et al. (2013) show that young/small firms were hit hard in the Great Recession due to the collapse in housing prices and thus their collateral. Laeven and Valencia (2013) investigates the importance of supply-side credit market frictions, and show that financially dependent firms are positively affected by bank recapitalization.
Our model generates firm dynamics that are consistent with empirical regularities, and contributes to the theoretical strand of the literature. Cooley and Quadrini (2001) develop a theoretical framework with asymmetric information and costly state verification to investigate the importance of financial frictions and persistent shocks on firm dynamics. The lending relationships between financial intermediaries and firms are limited to one-period debt contracts. As a result, the lending policy is not history-of-firm-performance-dependent and risk-sharing between firms and banks is restricted. In an economy where creditors and firms form long-term lending relationships to overcome informational asymmetries, the implications for firm dynamics and macroeconomic aggregates can be very different. Albuquerque and Hopenhayn (2004), Clementi and Hopenhayn (2006), and DeMarzo and Fishman (2007) study, in a partial equilibrium, firm dynamics determined by a lifetime-lending relationship with limited enforcement. Our paper extends their model into a general equilibrium framework. This allows us to analyze the impact of a change in aggregate credit supply on firm dynamics, which works through the equilibrium price effect on lending policy and on households’ occupational choices.

Finally, this paper adds to the literature on dynamic contracting under asymmetric information. The analysis of repeated games with informational asymmetries was initiated by Radner (1985) and Rogerson (1985). Green (1987), Thomas and Worrall (1990), and Atkeson and Lucas (1992) extend the previous framework into an infinite horizon and formulate the problem recursively with incentive-compatibility constraints. The resulting dynamic contracts display a risk-sharing nature between risk-neutral banks and risk-averse entrepreneurs who are exposed to idiosyncratic productivity shocks. Our analysis, and those of the others cited, feature financial friction (i.e., firms’ borrowing constraint) as a result of asymmetric information. This is in contrast to the other strand of literature that connects financial friction to the issue of collateral (Clementi and Hopenhayn, 2006; Verani, 2015) or investigates dynamic contracting in an environment with the possibility

\[^{10}\text{For instance, Smith and Wang (2006) show that, “relative to dynamic contracting, one-period debt contacts lower the equilibrium fraction of entrepreneurs, thereby reducing equilibrium output. Moreover, this reduction in output becomes larger as the monitoring cost increases.”}\]
of auditing (Verani, 2015) or with limited enforcement (Albuquerque and Hopenhayn, 2004).

The structure of the paper is as follows: Section 2 introduces the theoretical model. Therein, the workers’ and entrepreneurs’ problems and the role of financial intermediaries are described. Further, the recursive formulation of the dynamic lending contracts is presented and theoretical properties are discussed. Section 3 provides the aggregation and equilibrium conditions and defines the stationary general equilibrium. In Section 4 we calibrate the model and solve for the equilibrium numerically. The optimal dynamic contract characterizes the distribution of firms and firm dynamics in the economy. In Section 5 we investigate how a change in aggregate credit supply influences firms and the macroeconomic aggregates. Section 6 states our conclusions.

2 A model of firm dynamics in general equilibrium

2.1 Model set-up

Consider an infinite time horizon model with finite life expectancy. At the beginning of each period, an exogenous mass $1 - \Delta$ of \emph{ex ante} identical households are born. A household survives to the next period with probability $\Delta$. Therefore, the population size of households of age $\tau = 0, 1, \ldots$ is $(1 - \Delta)\Delta^{\tau}$.

Each newborn household decides to become a worker or an entrepreneur right after birth. The occupational choices are irreversible over lifetime. Denote the share of entrepreneurs in households of age $\tau$ at time $t$ by $\lambda^*_t$. $\lambda^*_t$ is determined endogenously in equilibrium by the labor market clearing condition (see (28)). A worker supplies labor, consumes and saves part of its income. An entrepreneur runs a firm which uses labor and capital as

\footnote{In the stationary equilibrium (see the definition in Section 3.3), where there are households of all ages ($\tau = 0, 1, \ldots$) in the economy, the total population size is constant at 1.}
inputs and consumes entrepreneurial income (net revenue from production). In addition to the households, there are banks which act as financial intermediaries between workers and entrepreneurs. Namely, they take annuity deposits from workers and offer capital and financing contracts to the entrepreneurs for their production.

2.2 Households

Households (both workers and entrepreneurs) are endowed with one unit of labor each period and no wealth at birth. The instantaneous utility function of the households is $U(c, l)$, where $c$ is the consumption level and $l$ is the labor supply. $U(c, l)$ is decreasing in $l$ and increasing, strictly concave and bounded in $c$. Households discount future with rate $\beta$.

2.2.1 Workers

In each period, workers supply labor for production and get wage income in return. Wage income as well as wealth can be used for consumption of final goods or as savings for wealth (and thus consumption) in future periods in the form of one-period annuity deposits in the banks. In each period $t$, the workers of age $\tau$ buy at the end of the period $A_{\tau,t+1} \geq 0$ units of the annuity at price $p_t^A$. This entitles the worker to receive wealth level $A_{\tau,t+1}$ in period $t+1$ conditional on survival. The annuity deposits are priced competitively (actuarially fair) such that banks make zero profit from offering them to the workers. Thus, the price is given by

$$p_t^A = \frac{\Delta}{1 + r_{t+1}},$$

(1)

where $r_{t+1}$ is the interest rate in $t + 1$ and is endogenously determined in equilibrium (see Section 3.2). Therefore, using current period deposit level $A \geq 0$ as state variable, the

\footnote{See appendix A.1 for a detailed derivation of the actuarially fair price.}
worker’s problem can be formulated recursively as

\[ V^W(A; r, w) = \max_{c, l, A'} \left\{ U(c, l) + \Delta \beta V^W(A'; r', w') \right\} , \tag{2} \]

subject to

\[ c + p^A A' = w l + A, \tag{3} \]
\[ c \geq 0, \ l \in [0, 1], \ A' \geq 0. \]

\( V^W(A; r, w) \) is the worker’s value function (i.e., continuation utility) given today’s wealth level \( A \), interest rate \( r \) and wage rate \( w \). \( p^A \) is given by \( (1) \). A prime indicates variables of tomorrow. \( \beta \) is the future discount rate, and \( \Delta \) captures the probability that the worker survives to the next period. Denote the policy function of optimal saving \( A' \) and labor choice \( l \), respectively, by

\[ A' = g(A; r', w), \quad l' = h(A; r', w). \tag{4} \]

### 2.2.2 Entrepreneurs and firms

Each entrepreneur runs a firm. They supply entrepreneurial labor and derive utility from consumption of net revenue from production. Entrepreneurs and firms are associated for the whole lifetime. Namely, a newborn household who becomes entrepreneur opens a firm and runs the firm for the entire lifetime until death; then the firm exits the market. Thus, the firm’s exit rate is exogenously given by the household’s death rate \( 1 - \Delta \).

Firms produce a single output (numéraire) using capital from banks \( k \), and labor from workers \( l \). The output can be consumed directly or used as capital. In addition, a fixed amount of entrepreneurial labor \( L^E \) is needed for setting up / managing the production. The production requires capital \( k \) and labor from workers \( l \), and takes the following form:

\[ Y(k_t, l_t) = \theta_t F(k_t, l_t), \]
where $F(\cdot)$ reflects the production technology that transforms capital and labor inputs into the final product. It exhibits decreasing returns to scale. We assume the function to be continuous and strictly concave.

The level of $\theta_t$ represents the productivity at time $t$. In each period, the productivity is subject to an idiosyncratic shock with state space $S = \{1, 2, \ldots, S\}$ and the corresponding realization of states, $\theta_t \in \Theta = \{\theta_1, \theta_2, \ldots, \theta_S\}$. The shock is i.i.d. over entrepreneurs and time. The probability distribution of the states is $\{\pi_s\}_{s \in S}$ with $\sum_{s \in S} \pi_s = 1$. Without loss of generality, let $\theta_i < \theta_j$ if $i < j$. At any time $t$, a firm of age $\tau$ has a history of $\tau$ productivity realizations, $\theta^\tau_t = (\theta_{t-\tau}, \ldots, \theta_{t-\tau+i}, \ldots, \theta_t), i \in \{0, 1, \ldots, \tau\}$. Heterogeneity among firms therefore emerges from two dimensions: Age and history of productivity realizations. We assume that the realization of productivity shock is private information to the entrepreneur.

Prior to production (i.e., before the idiosyncratic shock is realized), the entrepreneurs need to purchase capital and pay the workers. By assumption, the entrepreneurs are neither endowed with wealth nor do they accumulate wealth from their production revenues over lifetime. In other words, self-financing of production or inter-firm lending are excluded. Hence, they need external financing. We restrict the source of financing to bank loans. Bank loans and repayments arise from a lifetime financial contract between the bank and the entrepreneur. More specifically, the financial contract entitles the entrepreneurs each period to some amount of bank loans $b$, which is used to cover the

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13 We assume that banks can neither infer the productivity realization from output level, nor observe the information \textit{ex post}.

14 Given the assumption, entrepreneur’s problem is simplified to a one-period optimal employment of input factors (i.e., capital and labor). This assumption differs our model from the standard endogenous credit constraint model, where entrepreneurs accumulate saving and overcome financial constraint over lifetime. In our model, there is no intertemporal saving decision or decision on the optimal composition of equity and debt. See Quadrini (2009) for a detailed review of entrepreneurial financing.
production costs, and some repayments $m$ after production. For a given level of loans and factor prices, the entrepreneurs determine the optimal capital and labor employment by maximizing expected output. Remember that entrepreneurs do not make intertemporal savings decisions by assumption. The expected lifetime utility maximization is thus simplified to the following static problem:

$$\max_{k_t, l_t} \mathbb{E}(\theta_t) F(k_t, l_t)$$

subject to

$$w_t l_t + (r_t + \delta) k_t \leq b_t,$$

where $r_t$ is the interest rate in time $t$ and $\delta$ is the capital depreciation rate. We define

$$R(b_t; r_t, w_t) \equiv F(k^*_t, l^*_t)$$

with $k^*_t = k(b_t; r_t, w_t), l^*_t = l(b_t; r_t, w_t)$ being the solution to (5) at which the marginal rate of transformation correspond to the relative factor price of capital and labor. Notice that firms’ labor costs include only wage payments to workers. The implicit assumption is that the entrepreneurs do not supply the entrepreneurial labor $L^E$ in the labor market of workers. In what follows we denote the labor supply from workers as labor.

The entrepreneur’s consumption $c_t$ in each period is given by net revenue from production, which is gross production $\theta_t R(b_t)$ minus repayments to banks $m_t$:

$$c^E_t = \theta_t R(b_t; r_t, w_t) - m_t.$$  

An entrepreneur’s future expected utility starting from time $t$ (i.e., expected continuation utility) is given by

$$V^E_t = \sum_{i=t}^{\infty} (\beta \Delta)^{i-t} \mathbb{E}U(c^E_t, L^E_t),$$

where expectation is with respect to the history of productivity realizations (including the current period realization, $\theta_t$, which show up in $m_t$ as described in Section 2.5).}

\footnote{A detailed characterization of the financial contract, which includes bank loans $b$ as well as repayments $m$ is given in Section 2.5.}
According to the properties of the utility function, natural bounds for $V_t^E$ are given by $V_{min}^E$ and $V_{max}^E$, where

\[ V_{min}^E = \lim_{c \to 0} \frac{1}{1 - \beta \Delta} U(c, L^E) \quad \text{and} \quad V_{max}^E = \lim_{c \to \infty} \frac{1}{1 - \beta \Delta} U(c, L^E). \]  

In recursive formulation, the continuation utility of an entrepreneur at time $t$ can be written as follows according to (8):

\[ V_t^E = \mathbb{E} \left[ U(c_t^E, L^E) + \beta \Delta V_{t+1}^E \right]. \]  

2.3 Financial intermediaries

The banks serve as financial intermediaries between households and firms. Namely, they take annuity deposits from workers and offer capital for production to entrepreneurs. The credit relation between banks and entrepreneurs is characterized by a take-it-or-leave-it lifetime binding financial contract, which is offered to each newborn entrepreneur. We assume that both banks and entrepreneurs are fully committed to the contract in all possible future contingencies.\(^{16}\)

Banks are risk neutral profit maximizers and discount future at the current interest rate. There is free entry into the banking sector. Therefore, in equilibrium banks expect zero profits from each lending contract and size and ownership of the banks do not matter. Without loss of generality, we assume there exists a representative bank providing financial contracts to entrepreneurs of all ages and with all heterogeneous histories of productivity realizations.

The dynamic contract consists of bank loans to the entrepreneurs and repayments to the banks. Loans are advanced before production. Given the bank loans, firms decide the optimal demand for capital and for labor under uncertainty as described in (5). Banks and workers supply the input factors, respectively. And the firms pay the corresponding

\(^{16}\)Clementi and Hopenhayn (2006) where liquidation of firms after a sequence of bad shocks.
costs. After production taking place and productivity state being realized, entrepreneurs make repayments to banks according to their report of the current period productivity realization. Before formally characterizing the dynamic contract, we describe the timing of the model.

2.4 Timing

1. A mass of $1 - \Delta$ households is born. Each newborn household makes a lifetime irreversible occupational decision (to become an entrepreneur or a worker). The new entrepreneurs sign a lifetime binding financial contract with the banks.

2. The banks give loans $b(V^E)$ to all entrepreneurs in the economy, according to their promised value $V^E$ (defined in equation (10)). The amount is entitled by the respective terms of the contract (defined in Section 2.5).

3. Production takes place under uncertainty, using capital from banks and labor from workers as inputs. The costs of capital and of labor inputs are paid with the loans. Banks pay survived workers from last period their purchased annuity.

4. After production, entrepreneurs observe their productivity realization and report it to the bank. Then, entrepreneurs make state-contingent repayments $m_s(V^E)$ to the banks according to the financial contract and fully consume the remaining net production revenue. Further, the contract determines state-contingent promised values $V_s^E(V^E)$ as future state variable. Workers consume and purchase annuities for next period from the banks with their labor income and capital returns.

5. A share $1 - \Delta$ of the workers and entrepreneurs dies and the associated firms exit.

Figure 1 summarizes specifically the timing of the dynamic financial contract in one period.
2.5 Dynamic lending contract

Following the standard dynamic contracting model (e.g., Thomas and Worrall (1990), Atkeson and Lucas (1992)), the optimal dynamic contract is formulated recursively using the expected continuation utility $V_t^E$ (defined in (8)) – generally called the promised value – as the state variable. In this way, instead of tracking the entire productivity history, banks record firms’ promised value and formulate the terms of contracts as functions of the promised value. Therefore, for given interest rate and wage rate, $(r, w)$, the optimal contract can be determined by the following program:

$$P(V^E; r, w) = \max_{b, \{m_s, V^E_s\}_{s \in S}} \left[ -b + \sum_{s \in S} \pi_s \left[ m_s + \frac{\Delta}{1+r} P(V^E_{s+1}; r', w') \right] \right]$$

subject to

$$V^E = \sum_{s \in S} \pi_s [U (\theta_s R(b; r, w) - m_s, L^E) + \beta \Delta V^E_s],$$

$$U (\theta_i R(b; r, w) - m_i, L^E) + \beta \Delta V^E_i \geq U (\theta_j R(b; r, w) - m_j, L^E) + \beta \Delta V^E_j, \ \forall i, j \in S,$$

$$m_s \leq \theta_s R(b; r, w), \forall s \in S,$$

$$V^E_s \in [V^E_{\min}, V^E_{\max}].$$

$P(V^E; r, w)$ is the bank’s expected profit (value function) from a financial contract with state variable $V^E$ given $r$ and $w$. $b$ denotes the bank loans to firms. $\{m_s, V^E_s\}_{s \in S}$ are state-contingent repayments and future promised values, respectively. $\frac{1}{1+r}$ is banks’ future discount rate and $\Delta$ captures the fact that the entrepreneur survives to the next period.
with probability $\Delta$. $V_{min}^E$ and $V_{max}^E$ are the lower and upper bounds of the promised values defined in (9), respectively.

(PK) is the promise keeping constraint. It indicates that the terms of the contract must be such that the expected utility from today’s cash flows plus future promised values fulfill today’s promised value $V^E$.

(IC) ensures that a contract is incentive compatible. Specifically, it guarantees that entrepreneurs’ expected utility from truthfully reporting the realized state is higher than misreporting, and thus eliminates incentives to misreport.\(^{17}\)

The constraints (LL) stand for limited liability. Since by assumption entrepreneurs do not own wealth, their liability for repayments to the bank are limited by the extent of the production revenue (i.e., realized productivity shock times the production level corresponding to the bank loan level). Hence, a contract is feasible if the terms of the contract are such that the entrepreneurs consume a non-negative amount of the final products after any productivity realization.

The credibility constraint (CC) imposes that banks could only promise utility values that are achievable with non-negative finite cash flows; otherwise, the promised value would only be granted by violating (LL) sometime in the future or is never satisfiable, respectively. More precisely, (CC) captures that banks can never promise (i) less utility than achievable by non-negative consumption for all future periods or (ii) more utility than by infinite consumption for all future periods.

Formally, we define an optimal financial contract as follows:

**Definition 1.** For a given path $\{r_t, w_t\}_{t=0}^\infty$, the optimal dynamic contract is a sequence of functions $\{b(V_t^E; r_t, w_t), m(V_t^E, \theta_t; r_t, w_t), V^E(V_t^E, \theta_t; r_t, w_t)\}_{t=0}^\infty$ that solves program (A.4).

\(^{17}\)Note that according to the revelation principle, any equilibrium outcome can be achieved by a truth-telling mechanism. In particular, by imposing incentive constraints we can guarantee that entrepreneurs always report the actual realization of productivity $\theta_t$. Therefore, we focus only on truth-telling contracts.
Notice that bank loans, \( b(V_t^E; r_t, w_t) \), are made prior to production, whereas the repayments and future promised values, \( \{ m(V_t^E, \theta_t; r_t, w_t), V^E_t(V_t^E, \theta_t; r_t, w_t) \} \), are assigned afterwards. This implies that loans are only contingent on today’s promised value, whereas repayments are a function of today’s promised value and the reported productivity level including time \( \theta_t \). In addition, \( V_{t+1}^E = V^E(V_t^E, \theta_t; r_t, w_t) \) is the transition function of the state variable. We see directly from the contract that firms with the same promised value of today are assigned the same terms of contracts (independent of time \( t \) or age \( \tau \)). The promised value \( V^E \) can be used as an indicator of firms in the equilibrium analysis (see Section \[3.1\] for aggregation of entrepreneurs).

For notational simplicity we suppress the factor prices \( \{ r_t, w_t \} \) in the terms of contracts \( \{ b(V_t^E), m(V_t^E, \theta_t), V^E_t(V_t^E, \theta_t) \}_{t=0}^{\infty} \) and in the value function \( P(V^E) \) whenever it is not misleading.

### 2.5.1 Theoretical properties

In this section, we characterize the theoretical properties of the financial contracts defined in \[A.4\], and the implications of the optimal financial contract on the firms’ credit availability. Proposition 1 and 2 and Lemma 1 and 2 are standard results from the literature on dynamic contracts under asymmetric information.\(^\text{18}\)

The following proposition defines the necessary condition of an incentive compatible contract:

**Proposition 1.** Let \( \theta_s > \theta_{s-1}, \forall s \in S \). An incentive compatible contract satisfies \( m_s \geq m_{s-1}, c_s \geq c_{s-1} \) and \( V_s^E \geq V_{s-1}^E \). If \( m_s = m_{s-1}, V_s^E = V_{s-1}^E \), and vice versa.

**Proof.** See Appendix \[A.2.1\].

\(^\text{18}\)For example, see the Chapter on Optimal Social Insurance in \[Ljungqvist and Sargent (2000)\], which is based on the \[Thomas and Worrall (1999)\].
A financial intermediary requires more repayments after a high productivity shock, but delivers a high future promised value as well, and vice versa. In other words, banks postponing rewards for reporting high productivity realization, in order to induce truth-telling behavior of entrepreneurs. In this way, the financial contract provides risk averse firms an insurance (i.e., consumption smoothing) over the idiosyncratic shock. Entrepreneurs’ consumption is higher after a good productivity shock than after a bad one. This reflects that the optimal financial contract is a constrained optimal. Namely, the insurance provided through the financial contract is imperfect (i.e., entrepreneur consumption fluctuates over states) due to asymmetric information. Similar results also emerge from the optimal social insurance under asymmetric information studied by Thomas and Worrall (1990).

Define the incentive constraints for all \(i, j \in S\) as:

\[
C_{i,j} \equiv U(\theta_i R(b) - m_i, L^E) + \beta \Delta V^E_i - U(\theta_i R(b) - m_j, L^E) - \beta \Delta V^E_j \geq 0,
\]

where \(i\) is the actual state and \(j\) is the reported state. Then, the set of incentive constraints can be simplified with the following lemma.

**Lemma 1.** If the local downward constraints, \(C_{s,s-1} \geq 0\), and the local upward constraints, \(C_{s,s+1} \geq 0\), hold for each \(s \in S\), then the constraints \(C_{i,j} \geq 0\) hold \(\forall i, j \in S\).

*Proof.* See Appendix A.2.2. \(\square\)

Suppose for the following lemma and Proposition 2 and 3 that \(P(V^E)\) is strictly concave – a fact which is observed in the numerics. Using the concavity of the value function and Lemma 1, we can simplify the (IC) constraint that defined the optimal contract.

**Lemma 2.** For strictly concave \(P(V^E)\), for all states \(s \in S\), the optimal contract implies that the local downward constraints \(C_{s,s-1} \geq 0\) always bind, whereas the local upward constraints \(C_{s-1,s} \geq 0\) never bind for \(m_s > m_{s-1}\).
Proof. See Appendix A.2.3

In other words, entrepreneurs have no incentive to report a higher productivity than what they experienced. When repayment and future promised values are independent of their report (i.e., \( m_s = m_{s-1}, V_s^E = V_{s-1}^E \)), entrepreneurs are indifferent in their reporting strategy.

In addition, the optimal contract has the property of risk sharing between the entrepreneurs and the banks:

**Proposition 2.** For strictly concave \( P(V^E) \), both the entrepreneurs’ utility and the banks’ profits are non-decreasing with a higher productivity realization, that is: Under an optimal contract, for \( \theta_i > \theta_j \),

\[
U(\theta_i R(b) - m_i, L^E) + \beta \Delta V_i^E \geq U(\theta_j R(b) - m_j, L^E) + \beta \Delta V_j^E, \tag{13}
\]

\[
-b + m_i + \frac{\Delta}{1 + r} P(V_i^E) \geq -b + m_j + \frac{\Delta}{1 + r} P(V_j^E). \tag{14}
\]

Proof. See Appendix A.2.4.

The banks share part of the entrepreneurs’ productivity shock by asking for a low repayment after a low productivity realization. They benefit from high productivity shocks by extracting more repayments from the firms.

Next, we introduce the efficient level of bank loan, \( b^* \), which is implicitly determined by

\[
\mathbb{E}(\theta) R'(b^*; r, w) = 1, \tag{15}
\]

that is, marginal productivity equals marginal costs of one more unit of bank loans. Notice that the efficient level of bank loans corresponds to the optimal firm size if banks were the firm owners and observe the realization of the state directly.

Suppose for Proposition 3 that there are only two states in the state space, \( S = \{l, h\} \) with \( \theta_h > \theta_l \).
Proposition 3. For strictly concave $P(V^E)$, the optimal level of bank loans from the contract is not larger than the efficient level.

Proof. See Appendix A.2.5.

The intuition is as follows: When there is no asymmetric information, the optimal level of bank loans is determined by a tradeoff between the marginal cost of given loans and the expected revenue from additional investments. However, the presence of informational asymmetry induces additionally a marginal cost of maintaining incentives for firms. The higher marginal costs drives down the level of bank loans banks give. Therefore, the financial markets are endogenously incomplete due to asymmetric information, and financial constraints emerge from the optimal financial contract.

3 Aggregation and general equilibrium

So far we have characterized the optimization problems of the agents in the economy. More specifically, for a given sequence of factor prices $\{r_t, w_t\}_{t=0}^\infty$ and the share of entrepreneurs of each cohort $\{\lambda_t\}_{t=0}^\infty$, we get: (i) the workers’ optimal path of consumption, wealth accumulation and labor supply from (2); (ii) the entrepreneurs’ optimal path of capital and labor employment from (3); and (iii) the banks’ optimal path of terms of contracts with loans, repayments and future promised values from (A.4). Given the technical complexities, for combining the three partial parts to get the general equilibrium we focus on the stationary case with constant factor prices $\{r, w\}$ and a constant share of entrepreneurs $\lambda$. Specifically, we consider the age-dependent, time-independent supplies and demands of labor and capital, consumption of workers and entrepreneurs, bank loans and repayments. The equilibrium is then the prices $\{r, w\}$ and the share of entrepreneurs.

19A detailed discussion of the intuition combined with the proof of the proposition is given in Appendix A.2.5.
\( \lambda \) such that goods, labor and capital markets clear and banks make zero profit.

In this section, we first derive the macroeconomic aggregates (i.e., aggregate demand and aggregate supply of labor, of capital and of goods) by summing over the optimal decisions over all workers and all entrepreneurs, respectively. Using these aggregate variables, we write down the market clearing conditions, which determine the equilibrium factor prices and the share of entrepreneurs. Finally, we defined the stationary equilibrium formally in Section 3.3.

### 3.1 Aggregation

Workers are identical in their lifetime decisions within a cohort. Therefore, the aggregation of consumption \( C^W \), deposits \( D \) and labor supply \( L^S \) are over workers of different ages, which are given respectively as the following:

\[
C^W(r, w) = (1 - \lambda) \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau c(A_{\tau}, r, w) \tag{16}
\]

\[
D(r, w) = (1 - \lambda) \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau p^A A_{\tau+1} = (1 - \lambda) \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau p^A g(A_{\tau}, r, w) \tag{17}
\]

\[
L^S(r, w) = (1 - \lambda) \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau l_{\tau} = (1 - \lambda) \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau h(A_{\tau}, r, w) \tag{18}
\]

where \( 1 - \lambda \) is the endogenous share – and the mass, since the total population is normalized to 1 – of workers in the economy. \( (1 - \Delta) \Delta^\tau \) denotes the share of workers of age \( \tau = 0, 1, \ldots \), \( c(\cdot) \) is worker’s current period consumption given by (3), \( g(\cdot) \) and \( h(\cdot) \) are their saving and labor supply, respectively, defined in (4).

Entrepreneurs are heterogeneous in two dimensions: Age and history of productivity realizations. As is mentioned in Section 2.5, the promised value of firms summarizes all information on the history of productivity realizations. Therefore, the aggregation of entrepreneurs is over different promised values within cohort and over different ages. Denote the distribution of promised values of entrepreneurs within cohort \( \tau \) by \( \Psi_{\tau}(V^E) \),
\( \tau = 0, 1, \ldots \). We show in appendix A.3 that given the exogenous survival rate and the idiosyncratic shocks, the distribution of entrepreneurs’ promised values is stationary. In addition, we describe the numerical procedure of calculating the distribution function \( \Psi_\tau(V^E) \) in appendix C.3.

Following the optimal financial contract, promised values, \( V^E \), map uniquely into the level of bank loans and state-contingent repayments \( \{b(V^E; r, w), m(V^E, \theta_s; r, w)\} \). For given bank loans, firms’ optimal capital and labor employment, \( \{k^*(V^E; r, w), l^*(V^E; r, w)\} \), are the determined in (5). The aggregate bank loans \( B \), capital \( K^D \) and labor demand \( L^D \) are given as follows:

\[
B(r, w) = \lambda \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau \int b(V^E; r, w) d\Psi_\tau(V^E), \tag{19}
\]

\[
K^D(r, w) = \lambda \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau \int k^*(V^E; r, w) d\Psi_\tau(V^E), \tag{20}
\]

\[
L^D(r, w) = \lambda \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau \int l^*(V^E; r, w) d\Psi_\tau(V^E), \tag{21}
\]

where \( \lambda \) is the endogenously determined share of entrepreneurs in the economy. \( (1 - \Delta) \Delta^\tau \) is the share of entrepreneurs of age \( \tau = 0, 1, \ldots \).

Furthermore, the expected aggregate repayments from all entrepreneurs to banks are given by:

\[
M(r, w) = \lambda \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau \sum_{s \in S} \pi_s \int m(V^E, \theta_s; r, w) d\Psi_\tau(V^E). \tag{22}
\]

Similarly, the expected aggregate output \( Y \) and the consumption of the entrepreneurs \( C^E \) are given by:

\[
Y(r, w) = \lambda \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau \sum_{s \in S} \pi_s \int \theta_s R(b(V^E; r, w); r, w) d\Psi_\tau(V^E), \tag{23}
\]

\[
C^E(r, w) = \lambda \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau \sum_{s \in S} \pi_s \int c(V^E, \theta_s; r, w) d\Psi_\tau(V^E), \tag{24}
\]

where \( R(\cdot) \) is defined in (6) and \( c(\cdot) \) in (7).
Finally, banks’ equity is the accumulated retained earnings from the flows of bank loans and repayments. In a stationary equilibrium, it is determined by

\[ E(r, w) = (1 + r)E(r, w) + M(r, w) - B(r, w), \]

where \( E(r, w) \) denotes the bank equity, \((1 + r)E(r, w)\) are the gross returns on previous equity and \( M(r, w) - B(r, w) \) are the net aggregate payments.\(^{20}\) Rewriting the above equation, we have

\[ E(r, w) = \frac{B(r, w) - M(r, w)}{r}. \tag{25} \]

### 3.2 Equilibrium conditions

Having the aggregate variables, we can derive the equilibrium conditions. Newborn households are indifferent with respect to their occupational choices in equilibrium. Therefore, the expected lifetime utility of becoming a worker equals that of becoming an entrepreneur. Formally, we have:

\[ V^E_0 = V^W(0; r, w), \tag{26} \]

where \( V^W(0; r, w) \) is defined in (2) and \( V^E_0 \) is the promised value of an entrepreneur at the beginning of their lifetime.

In addition, the banking sector is competitive. Thus, banks make zero profit in expectation from each newly-signed contract in equilibrium, \( P(V^E_0; r, w) = 0 \). Under the indifferent-occupational-choice condition in (26), the zero-profit condition for banks is

\(^{20}\)Notice that this condition indicates that in the stationary equilibrium banks give on aggregate more loans than repayments they ask for; with the gap between \( B \) and \( M \) being exactly coverable by the interest from banks’ equity. Thus, the level of equity is endogenously kept constant in the stationary case. Since we do not characterize the path of how the economy converges to the stationary equilibrium, we cannot show numerically how the accumulation of banks’ equity converges to the stationary equilibrium level. However, we give in Appendix E a non-rigorous intuition of how an economy may evolve from the very beginning of time to the stationary equilibrium.
reformulated as

\[ P(V^W(0; r, w); r, w) = 0. \]  \hspace{1cm} (27)

Finally, labor, capital and goods markets clear. Labor market clearing requires that aggregate labor supply from workers equals aggregate demand for labor by the entrepreneurs. This is

\[ L^D(r, w) = L^S(r, w), \]  \hspace{1cm} (28)

with \( L^S(r, w) \) and \( L^D(r, w) \) defined in (18) and (21), respectively.

Capital supply in the economy consists of aggregate deposits from the workers plus banks’ equity. The capital market clearing is thus given by:

\[ K^S(r, w) \equiv D(r, w) + E(r, w) = K^D(r, w), \]  \hspace{1cm} (29)

where \( D(r, w) \), \( E(r, w) \) and \( K^D(r, w) \) are given in (17), (25) and (20), respectively.

The goods market is cleared if aggregate output equals the sum of households’ consumption plus aggregate investments, where the latter is equal to depreciated capital in a stationary equilibrium. Formally, the condition is

\[ Y(r, w) = C^W(r, w) + C^E(r, w) + \delta K^D(r, w). \]  \hspace{1cm} (30)

It is directly implied by the labor and the capital market clearing conditions, (28) and (29), as is shown in Appendix A.4.

### 3.3 Definition of general equilibrium

With the agents’ optimal behavior derived from the respective optimization problems and the general equilibrium conditions, we can now define the stationary general equilibrium in the economy.

**Definition 2.** A stationary general equilibrium is characterized by a stationary distribution of workers of different ages, and the corresponding capital and labor supply
The factor prices \((r, w)\) and share of entrepreneurs \(\lambda\) are such that,

1. labor, capital and goods market clear according to (28), (29) and (30).

2. banks make zero profit in expectation according to (27).

Given the complexity of the problem, we determine the stationary equilibrium numerically. We do not deliver an analytical proof for the existence and the uniqueness of the stationary equilibrium globally. Nevertheless, we describe in detail the algorithm to find the stationary equilibrium in Appendix C.3. In addition, we provide sufficient conditions such that a locally unique stationary equilibrium exists.

4 Calibration and numerical results

In this section, we describe the quantitative analysis of the model. We illustrate the optimal dynamic contract and the distribution of firm size in equilibrium. In addition,
we show the main results on firm dynamics – development of firm size, firm growth and
the volatility of growth over lifetime.

The households’ utility function (workers and entrepreneurs) is given by

\[ U(c, l) = -\exp(-\gamma c) - \eta l^2, \quad \gamma, \eta > 0. \] (31)

It includes a CARA-part for consumption with \( \gamma \) being the absolute risk aversion and a
parabola part for the disutility of labor supply. The form of the utility function gives us
computational simplicity.

Firms produce with decreasing-return-to-scale production function:

\[ Y(k, l) = \theta_s \bar{a}^\alpha k^\alpha l^\alpha, \] (32)

where \( \theta_s \) denotes the state-dependent productivity realization, \( \bar{a} \) scales total factor pro-
ductivity, and \( \alpha_k \) and \( \alpha_l \) are the share of capital and labor, respectively. We simplify
the state space of firm productivity to two states: “high” and “low”, \( s \in \{ h, l \} \). Let the
corresponding productivity be given by \( \theta_h = \theta + \sigma \) and \( \theta_l = \theta - \sigma \), \( \sigma > 0 \), and probability
by \( \pi_h \) and \( \pi_l \), respectively.

We summarize the value of exogenous parameters in Table 1. One period in the model
corresponds to one year in reality. The survival rate \( \Delta = 0.92 \) is chosen such that
the death rate \( 1 - \Delta \) corresponds approximately to the yearly exit rate of firms in the
US manufacturing, 8\%. The discount rate \( \beta \) is within the range of standard values in
literature. Household preference parameter \( \gamma \) and \( \eta \) are internally calibrated such that
workers’ labor supply is about 30% of their labor endowment. Further, we assume both
states are equally likely. Then, the values of \( \theta_h \) and \( \theta_l \) imply an expected productivity
realization of \( \theta = 1 \), with standard deviation of 0.25. \( L^E \) corresponds to a third of
an entrepreneur’s labor endowment. \( \alpha_k \) and \( \alpha_l \) correspond to the capital and labor
shares of output. The depreciation rate \( \delta = 0.1 \) corresponds to a common number in
literature reflecting a quarterly depreciation rate of approximately 2.5\%. The assumed
utility function and the parameter values determine the boundaries of the promised value, 
\( V_{\text{min}}^E = -9.26 \) and \( V_{\text{max}}^E = -0.49 \) given by [3].

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Survival rate ( \Delta )</td>
<td>0.92</td>
</tr>
<tr>
<td>Discount rate ( \beta )</td>
<td>0.963</td>
</tr>
<tr>
<td>Household preferences ( \gamma )</td>
<td>2</td>
</tr>
<tr>
<td>( \eta )</td>
<td>0.5</td>
</tr>
<tr>
<td>Probability of bad state ( \pi_l )</td>
<td>0.5</td>
</tr>
<tr>
<td>High productivity ( \theta_h )</td>
<td>1.25</td>
</tr>
<tr>
<td>Low productivity ( \theta_l )</td>
<td>0.75</td>
</tr>
<tr>
<td>Fixed entrepreneur labor ( L^E )</td>
<td>1/3</td>
</tr>
<tr>
<td>Share of capital ( \alpha_k )</td>
<td>0.35</td>
</tr>
<tr>
<td>Share of labor ( \alpha_l )</td>
<td>0.6</td>
</tr>
<tr>
<td>Productivity scale ( \bar{a} )</td>
<td>1/3</td>
</tr>
<tr>
<td>Depreciation rate ( \delta )</td>
<td>0.1</td>
</tr>
</tbody>
</table>

### 4.1 Optimal dynamic contract

In this section, we illustrate the banks’ optimal dynamic contract for given factor prices, \( \{r, w\} \)\(^{21}\). We discuss the qualitative properties and the implications for the endogenous credit constraint at firm level. Worker’s problem defined in (2) can be solved as in standard lifetime utility maximization problem. Properties of policy functions and value function follow. Given the production function in (32), the entrepreneurs’ capital

\(^{21}\)In the figures, we use \( w = 0.1599 \) and \( r = 0.0417 \), which are the equilibrium values later determined numerically in the general equilibrium in Section 4.2 by using the search algorithm described in Appendix C.4. For simplicity we suppress \( w \) and \( r \) in the notation.
and labor demand can be analytically solved. We show the solution to workers and entrepreneurs’ problem in Appendix B.

Figure 2 shows (for given \( r \) and \( w \)) as a function of today’s promised value \( V^E \) (state variable), the banks’ profit \( P(V^E) \), state-contingent future promised value \( V^E_s(V^E) \), state-contingent repayments \( m_s(V^E) \) and the bank loans \( b(V^E) \), \( s \in \{l, h\} \), with \( l \) and \( h \) standing for low and high productivity realizations, respectively.\(^{22}\)

The banks’ profit is \( P(V^E) \) is strictly concave. For \( V^E \) not close to \( V^E_{min} \), \( P(V^E) \) decreases in \( V^E \).\(^{23}\) In equilibrium, competitive banking sector implies zero expected profit from any newly signed contract. Therefore, \( P(V^E) = 0 \), pins down the initial promised value \( V^E_0 \)

\(^{22}\)See Appendix C.2 for the numerical procedure to solve the recursive formulated lending contract.

\(^{23}\)The increasing part of the value function reflects a larger feasible set defined by the constraints as \( V^E \) increases in the region close to \( V^E_{min} \). Notice that at \( V^E = V^E_{min} \), there is only one feasible contract.

Figure 2: Optimal contract
(i.e., initial lifetime expected utility of entrepreneurs). Under households’ indifferent-occupational-choice condition defined in (26), this equals to the lifetime expected utility of the workers, $V^W(0)$, in equilibrium.

The state-contingent future promised values, $V^E_t(V^E)$ and $V^E_h(V^E)$ (subplot 2), are strictly increasing in $V^E$. This together with firms’ productivity realization governs the development of firms’ promised value. Specifically, after a high productivity shock, an optimal financial contract implies higher promised value for next period, $V^E_h > V^E$, and after a low productivity shock, a low value $V^E_l < V^E$. Overall, this implies that firms that experience more good productivity shocks tend to have higher promised value, and vice versa.\footnote{An illustration of the firm productivity shock and the corresponding terms of contracts are given in Appendix F.1.}

In the quantitative analysis, firms promised utility increases on average as they get older, which implies an increase in firm size over lifetime.

State-contingent repayments $m_s(V^E), s \in \{h, l\}$ are non-monotonic; repayments $m_s(V^E)$ first increase in $V^E$ and then decreases at higher promised values.\footnote{Non-monotonicity can arise as a result of the functional forms of the utility, the production and the profit function, and their relative curvature compared to each other; the banks fulfill higher promised values $V^E$ by both higher future promised utility and higher current consumption (through $b$ to $m_s$).}

At low promised value, firms make positive repayments to financial intermediaries, whereas repayments are negative at high promised value. This property, in combination with an increasing average promised value as entrepreneurs get older, implies that entrepreneurs make deposits in banks when young and consume more when they get older. This indicates that the optimal dynamic contract is back-loaded with respects to entrepreneurs’ consumption.

As is discussed in Proposition 1, higher repayments after high productivity realization together with higher future promised value (i.e., $m_h > m_l$ and $V^E_h > V^E_l$) induces truth-telling incentives and partial insurance over idiosyncratic shocks.

that satisfies all constraints. Namely, $b = 0, m_s = 0, V^E_s = V_{min}, s \in \{h, l\}$. This is because entrepreneur consumption must be non-negative and banks cannot promise an expected utility, $V^E$, unachievable with a non-negative consumption flow (i.e., $V^E$ must satisfy $V^E \geq V_{min}^E$).

24An illustration of the firm productivity shock and the corresponding terms of contracts are given in Appendix F.1.
The level of lending $b(V^E)$ (subplot 4) is strictly increasing in $V^E$, and approaches the efficient level defined in (15) gradually as $V^E$ increases. The transition function of the promised value, $V_s^E(V^E)$, together with the lending function, $b(V^E)$, generates firm dynamics: For given current period productivity realization, the future promised value to entrepreneurs, $V_s^E(V^E)$, determines the level of tomorrow’s lending and thus the evolution of the firm size. Since firms’ average promised value increases overtime, the average lending is higher as firms get older. This implies that older firms are less financially constrained than younger ones. In addition, notice that $\Pi \equiv \sum_{s \in \{h,l\}} \pi_s m_s(V^E) - b(V^E)$ represents banks’ current period retain earnings from the contracts with promised value, $V^E$. Quantitative analysis shows that $\Pi$ is positive for low $V^E$ and becomes negative as $V^E$ increases (see Appendix E). This implies that under asymmetric information, banks retain earnings from firms that experience more bad shocks, and reward firms with good histories with the money. In the Appendix, we describe intuitively the accumulation of bank equity along the development of the economy using this property.

### 4.2 General equilibrium

In this section, we show the stationary equilibrium, as well as firm dynamics in the economy. Algorithm for solving the stationary equilibrium is described in Appendix C.4. The theoretical intuition for the existence and uniqueness of the stationary equilibrium locally is given in Appendix D.

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26 The efficient level $b^* = 1.1814$. The specific shape of $b(V^E)$ is the result of the functional forms of the utility and the production function and their relative curvature compared to each other (see A.10 in Appendix A.2). There are unstable $b(V^E)$ for $V^E$-values approaching $V_{max}^E$ (corresponding to positive infinite consumption in all periods) due to computational difficulties for values close to $V_{max}^E$. However, for determining the equilibrium this problem is negligible because firms hardly reach promised $V^E$-values in the region close to $V_{max}^E$ when starting at $V_0^E = -8.36$ as derived in the general equilibrium (e.g., 65 years of always high productivity shock, which would leads to $V^E > -1$ has probability $(\Delta(1-\pi_l))^{65} = 1.2 \cdot 10^{-22} \approx 0$). Further, the highest $V^E$ reached by an entrepreneur in the simulation of our economy is only $-1.51$. 

---
Table 2: Equilibrium parameter

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest rate $r^*$</td>
<td>4.17%</td>
</tr>
<tr>
<td>Wage $w^*$</td>
<td>0.1599</td>
</tr>
<tr>
<td>Share of entrepreneurs $\lambda^*$</td>
<td>7.62%</td>
</tr>
</tbody>
</table>

The interest rate in the calibrated economy is around 4.17%, which corresponds to the annual interest rate used commonly in literature. The equilibrium share of entrepreneurs is 7.62%, which is approximately the rate of self-employed labor in the U.S. over the last years (data from OECD).

Table 3: Equilibrium values

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lifetime utility</td>
<td>$V_0^E = V^W(0, r^<em>, w^</em>) = -8.3549$</td>
</tr>
<tr>
<td>Total labor supply</td>
<td>$L^S$</td>
</tr>
<tr>
<td>Total capital supply</td>
<td>$D$</td>
</tr>
<tr>
<td>Total bank loans</td>
<td>$B$</td>
</tr>
<tr>
<td>Total labor demand</td>
<td>$L^D$</td>
</tr>
<tr>
<td>Total capital demand</td>
<td>$K$</td>
</tr>
<tr>
<td>Total repayments</td>
<td>$M$</td>
</tr>
</tbody>
</table>

The equilibrium macroeconomic aggregates are given in Table 3. The lifetime utility of workers, and thus the initial promised utility of entrepreneurs, is $V_0^E = V^W(0, r^*, w^*) = -8.36$. The total labor supply $L^S$ corresponds to about a third of a worker’s labor endowment, which is in line with standard values from the empirics. Further, from the amount of bank loans $B$ and repayments $M$ given in Table 3 we can calculate the amount of banks’ equity $E$ using (25). This indicates is an equity ratio $E/K = 14.68%$. This number is above current levels in large international banks, but below the proposed level.
of 20% by Admati and Hellwig (2013). We will show in Section 5 how the equilibrium factor prices, the share of entrepreneurs, and the macroeconomic aggregates respond to a change in aggregate credit supply in the economy.

4.3 Firm distributions

In this equilibrium, we can derive distributions for firm characteristics from the simulation of the paths of the entrepreneurs’ lives. Figure 3 shows the distribution of entrepreneurs in the economy with respect to different characteristics: Age, promised values, repayments and bank loans.

Figure 3: Distribution of age, promised values, repayments and bank loans
Subplot (a) shows the distribution of entrepreneurs’ ages. With a share of $1 - \Delta = 8\%$ most entrepreneurs are newborns. Then, one-year old represent a share of $(1 - \Delta)\Delta = 7.36\%$ and so on. Finally, the share of entrepreneurs older than 50 years account for only $0.16\%$ in our economy.

Subplot (b) shows the distribution of promised values $V^E$. We get the histogram of the distribution of firm promised values $\Psi(V^E)$ as shown in Subplot (b) by counting the number of entrepreneurs in the economy in different bins of $V^E \in [V_{min}, V_{max}]$. The plot indicates clearly that the mass of the promised values lies around the starting value $V^E_0 = -8.36$. Firm heterogeneity then arises from the different length and composition of productivity realizations over firms’ lifetime. The further away from the starting value $V^E_0$, the lower is the density of $V^E$ because longer and more heterogeneous life paths underlie such values.

Subplot (c) shows the distribution of repayments. It follows directly from the distribution $\Psi(V^E)$ (because $V^E$ is the underlying state variable). Depending heavily on the current period productivity realization, the levels of repayments are separated into two groups. This means, the repayments exhibit two distinct sub-distributions because the difference in repayments of high and low state are relatively large (compare $m_h(V^E)$ and $m_l(V^E)$ in Figure 2).

Subplot (d) shows the distribution of bank loans. It also follows directly from the distribution $\Psi(V^E)$ (because $V^E$ is the underlying state variable). It captures the firm size distribution measured by the levels of bank loans. From Figure 2 follow that for many $V^E$ the optimal level of banks loans lies around the value $b(V^E) \approx 1.12$ (see relatively flat part in Figure 2). This means, many firms get such levels of banks loans so that the mode of the distribution of bank loans lies around this value. Thus, the negative skewness in the distribution of $b$ is the result of the less strongly increasing part of $b(V^E)$ seen in Figure 2.
4.4 Firm dynamics

By considering now firm distribution of different cohorts separately (i.e., all entrepreneurs of the same age \( \tau \)), the model allows us to get firm dynamics: Average firms’ size, growth and variance of growth at different ages.

First, using the simulation of life path of entrepreneurs in Section 4.2 we generate the distribution of promised values \( \Psi(VE) \) of entrepreneurs at different ages \( \tau \). The development of \( \Psi(VE) \) for selected cohorts with age \( \tau = \{1, 2, 4, 8, 16, 32, 64, 209\} \) is shown in Figure 4.

The newborns \( \tau = 1 \) are all identical with the same starting promised value \( V_0E = -8.36 \). Surviving firms then experience either high or low productivity realizations and are updated with higher or lower future promised value levels, respectively. Over time as \( \tau \) gets larger, histories of productivity realizations get more heterogeneous due to the i.i.d. shocks. The distribution of promised values, \( \Psi(VE) \), gets more dispersed. In

\[ \text{Figure 4: Development of entrepreneurs’ promised value distributions} \]

\[ \text{The newborns } \tau = 1 \text{ are all identical with the same starting promised value } V_0E = -8.36. \text{ Surviving firms then } \]

\[ \text{experience either high or low productivity realizations and are updated with higher or lower future promised value levels, respectively. Over time as } \tau \text{ gets larger, histories of productivity realizations get more heterogeneous due to the i.i.d. shocks. The distribution of promised values, } \Psi(VE), \text{ gets more dispersed. In} \]

---

\[ ^{27}\text{This maps directly into the distributions of bank loans and repayments. The corresponding distributions of } b \text{ and } m \text{ are shown in Figure 14 and 15 in Appendix F, respectively.} \]
addition, as age advances cohort size becomes smaller because firms have been exiting with the exogenous death rate $1 - \Delta$. Eventually, (almost) all firms of a given cohort exit the market so that the distribution $\Psi_\tau(V^E)$ of old cohorts consist of very few individual observations.

Following the cohort distribution, $\{\Psi_\tau(V^E)\}_{\tau=0}^\infty$, we can get firm dynamics such as average size, growth and variance of growth at different ages $\tau$ of entrepreneurs. Such firm dynamics are shown in Figure 5.

Figure 5 shows in Subplot 1 an increasing average size of firms at different ages. To see the trend of the firm dynamics more clearly, we plot the 5-year moving average (e.g., the value of the average size at age 10 is the weighted average size of firms with age 10-14.)

There is a decrease in firm size between the one-year-olds and the two-year-olds. To see why, first, notice that the starting promised value, $V^E_0 = -8.36$, is at the right end of the steep part of the $b(V^E)$-function; a low productivity shock lowers $b$ more than a high productivity shock increases $b$. In addition, since the history of productivity shocks is not very heterogeneous after one period (i.e., 50% are high and 50% are low), the decrease from the low productivity shock is directly reflected in the average size. For more periods the history of productivity shocks of entrepreneurs becomes more heterogeneous and the average is thus less dependent on the level of bank loans corresponding to a specific history of productivity realizations.
size is measured in terms of the level of banks loans. Hence, our model predicts a positive relation between firm size and their age, which is in line with empirical observations.

In Subplot 2 we plot firms’ average growth rates at different ages. We define the growth rate of a firm at age $\tau$ by the percentage change in bank loans relative to last period’s loan, $g_{\tau} \equiv \frac{b_{\tau}-b_{\tau-1}}{b_{\tau-1}}$, where $b_{\tau}$ and $b_{\tau-1}$ are bank loans of today and of yesterday, respectively. The average growth rate of all firms at age $\tau$ is measured by the mean of $g_{\tau}$ among all entrepreneurs in this cohort. The graph shows that firms’ average growth is positive, but the rate decreases with firm age. The same holds for the variance of the growth rate (i.e., the variance of $g_{\tau}$) which is shown in Subplot 3. This means that on average older firms grow less, but in a more stable way.

These patterns are also found by Clementi and Hopenhayn (2006), Gross and Verani (2013) and Verani (2015), and are observed in industry data (e.g., Evans (1987)). This suggests that empirical firm dynamics can be explained by the design of the optimal financial contracts with endogenous borrowing constraints.

5 The real effects of tighter aggregate capital supply

In this section, we investigate the impact of tighter aggregate capital supply on the macroeconomic aggregates, on the credit availability to firms, as well as on firm dynamics. We focus on the long-run effect by comparing the stationary equilibrium under different levels of reserve ratios.

30 Firm size can be equivalently measured by the level of capital employment or labor employment. This can be seen from the linear relation between $b$ and $k^*$, $b$ and $l^*$, defined in (B.1).
31 The observation discussed in footnote 29 is the reason for the outlier of the average growth (and also of the variance) in the first year. Note that the less smooth pattern for young firms comes from the fact that at the beginning firms have less different productivity paths, so that we have in this sense not enough cases of observations. The less smooth pattern for older firm arises since firms are dying and not many observations are left.
Suppose that for each unit of deposit, banks must hold a share \( \mu \) as reserves. Therefore, the capital market clearing condition defined in (29) becomes

\[
K^D \leq (1 - \mu)D + E.
\] (33)

In other words, the aggregate capital supply decreases as reserve ratio increases. The convenience of introducing the reserve ratio is that it only changes the capital market clearing condition, and thus the mechanism of the comparative static analysis is relatively transparent. As we will discuss in more detail in Section 5.1, 5.2 and 5.3, the outcomes are generated by an equilibrium price effect on banks’ lending policy, on households’ occupational choices, and on entrepreneurs’ optimal demand for capital and labor.

Using the parameter values given in Table 1, we calculate the equilibrium factor prices and the share of entrepreneurs at reserve ratio, \( \mu = \{0, 0.2, 0.4\} \). The quantitative results are summarized in Table 4. For the calibrated economy, a rise in the reserve ratio from zero to 20% leads to a 1.68% increase in the interest rate from 0.0417 to 0.0424, and a 1.87% decrease in the share of entrepreneurs from 0.0762 to 0.0757 in the long run.

| Table 4: Comparative statics of \( \mu \) on equilibrium variables |
|------------------|---|---|---|---|
|                  | \( \mu = 0 \) | \( \mu = 0.2 \) | \( \mu = 0.4 \) | Sign |
| Interest rate, \( r \) | 0.0417 | 0.0424 | 0.0436 | + |
| Wage rate, \( w \) | 0.1599 | 0.1594 | 0.1587 | - |
| Share of entrepreneurs, \( \lambda \) | 0.0762 | 0.0757 | 0.0748 | - |

Intuitively, as aggregate capital supply decreases, market clearing condition implies that the interest rate increases. Higher interest rate increases workers’ lifetime expected util-

\[32\] A convenient way of interpreting \( \mu \) is reserve requirements on banks. Nevertheless, our framework is applicable to other circumstances where exogenous shocks within the financial markets decrease the aggregate capital supply to firms.

\[33\] The outcomes for \( \mu = 0 \) coincide with the benchmark case in Section 5.1. \( \mu = 0.4 \) is a particularly large number in the context of reserve requirements or generally for aggregate capital shocks. We include the equilibrium outcomes for \( \mu = 0.4 \) to show the robustness of the results.
ity. The higher utility attracts newborn households to become workers and the share of entrepreneurs declines. On the other hand, higher interest rate increases the capital costs of firm production. Higher capital costs decreases firm profit for given level of bank loans and thus firms’ repayment to banks. Therefore, bank profit decreases. Competitive banks promise lower utility to newborn firms – according to zero-profit condition, which maps into lower bank loans and tighter financial constraints. The decline on wage rate is the result of counteracting effects. On the one hand, the contraction of bank credit lowers firms’ demand for labor on average. At the same time, the larger share of workers in equilibrium (i.e., $\lambda$ increases) increases labor supply. Lower demand and higher supply both push down wage rate. On the other hand, as the relative price of capital to labor increases, firms increase their labor intensity. In other words, relative demand for labor increases. Besides, the income effect of higher interest rate decreases workers’ labor supply. These two effects tend to drive up the wage rate. Therefore, the net effect of these forces matter. We show that the negative effect – of lower labor demand due to tighter financial constraint and of higher labor supply from larger share of workers – on wage rate dominates.

A more technical way of thinking the mechanism is to use bank’s zero profit condition and households’ indifferent occupational choice decision. Since banks’ profit decreases as the interest rate increases, the zero profit condition $P(V^E_0; r, w) = 0$ together with a decreasing profit function implies that $V^E_0$ decreases. However, as interest rate increases, workers’ lifetime expected utility, $V^W(0; r, w)$, increases. Households’ indifferent occupational choice decision indicates that in equilibrium, entrepreneurs’ initial promised utility must increase as well. To fulfill the equilibrium condition $P(V^W(0; r, w); r, w) = 0$, the wage rate must reconcile the two forces. This means it must decreases such that a new equilibrium is achieved.
5.1 Impact on the macroeconomic aggregates

Now we show that the equilibrium price effect on the macroeconomic aggregates. The results are summarized in Table 5. An increase of reserve ratio from zero to 20% decreases the aggregate capital supply and the aggregate labor supply by 1.20%. We can see this more clearly from the demand side, which equals to the supply in equilibrium. Higher reserve ratio implies lower level of bank loans and tighter financial constraints on firms. This decreases firms’ demand for capital and labor. In addition, the share of entrepreneurs decreases in equilibrium. Therefore, the two effects combined drives down the aggregate capital and aggregate labor demand. Notice that the change in equilibrium factor prices implies that firms produce at higher labor intensity in equilibrium. Numerical outcomes show that the average labor intensity (i.e., $L^D/K^D$) increases by 0.72% from 1.520 to 1.531. As a result of tighter financial constraints, the aggregate output decreases. This is further amplified by a lower share of entrepreneurs. In equilibrium, the aggregate output decreases by 0.70%. One interesting observation is that as aggregate capital supply decreases, banks equity and aggregate deposits from workers increase. Nevertheless, the aggregate capital supply decreases, because part of the deposits are saved as reserves.

Table 5: Comparative statics of $\mu$ on equilibrium variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>$\mu = 0$</th>
<th>$\mu = 0.2$</th>
<th>$\mu = 0.4$</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate output, $\lambda E(\theta)R(b)$</td>
<td>0.0719</td>
<td>0.0714</td>
<td>0.0706</td>
<td>-</td>
</tr>
<tr>
<td>Aggregate labor supply</td>
<td>0.2660</td>
<td>0.2647</td>
<td>0.2628</td>
<td>-</td>
</tr>
<tr>
<td>Aggregate capital supply</td>
<td>0.1750</td>
<td>0.1729</td>
<td>0.1694</td>
<td>-</td>
</tr>
<tr>
<td>Capital supply from workers, $(1 - \lambda)(1 - \mu)D$</td>
<td>0.1493</td>
<td>0.1464</td>
<td>0.1416</td>
<td>-</td>
</tr>
<tr>
<td>Capital supply from equity, $\lambda E$</td>
<td>0.0257</td>
<td>0.0265</td>
<td>0.0278</td>
<td>+</td>
</tr>
<tr>
<td>Equity ratio, $E/K^D$</td>
<td>0.147</td>
<td>0.153</td>
<td>0.164</td>
<td>+</td>
</tr>
</tbody>
</table>

\[^{35}\text{Since the size of the agents is normalized to 1, the aggregate value is the corresponding average value (see \((19-22)\)).}\]
5.2 Impact on firm dynamics

Subplot 1 shows the development of bank loans over lifetime under different reserve ratios, $\mu$. The equilibrium price effect and a lower initial promised values under high reserve ratios decrease the credit availability of young firms in the economy. However, average promised values increase as firms get older in an economy with higher reserve ratio. As a result, the size of old firms increases. The combination of the two gives the counter-clockwise shift of the average firm size as the reserve ratios increase.

Subplots 2 and 3 show that the average growth rate of firms and the variance of growth, respectively. As reserve ratios increase, both values decrease. This is mainly because the range of bank loans is smaller under a large reserve ratio. For a given change in the promised values, the change in the level of bank loans is lower. Therefore, the growth rate and variance of growth are smaller.

5.3 Reallocation of credit from young firms to old ones

The changes in interest rate and wage rate have counteracting effects on the optimal contract. Our quantitative results show that as the reserve ratio rises, the efficient size of
firms (measured by the level of bank loans, defined in (B.3)) decreases. The tighter reserve ratio raises the interest rate, and thus depresses availability of credit in the market.

In addition, the initial promised values of firms decreases as the reserve ratio increases. As a result, in particular new firms are more financially constrained. Furthermore, firms at older ages have higher average promised values in an economy with high reserve ratios. This tends to increase the credit availability of the old firms.

Table 6: Comparative statics of $\mu$ on firm dynamics

<table>
<thead>
<tr>
<th></th>
<th>$\mu = 0$</th>
<th>$\mu = 0.2$</th>
<th>$\mu = 0.4$</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average firm size, $b$</td>
<td>0.8836</td>
<td>0.8830</td>
<td>0.8832</td>
<td>-/+</td>
</tr>
<tr>
<td>Age group 1-20</td>
<td>0.8693</td>
<td>0.8681</td>
<td>0.8674</td>
<td>-</td>
</tr>
<tr>
<td>Age group 21-40</td>
<td>0.9381</td>
<td>0.9400</td>
<td>0.9439</td>
<td>+</td>
</tr>
<tr>
<td>Age group 41-60</td>
<td>0.9698</td>
<td>0.9727</td>
<td>0.9780</td>
<td>+</td>
</tr>
<tr>
<td>Average firm growth</td>
<td>0.01992</td>
<td>0.01988</td>
<td>0.01983</td>
<td>-</td>
</tr>
<tr>
<td>Variance of growth</td>
<td>0.058</td>
<td>0.057</td>
<td>0.055</td>
<td>-</td>
</tr>
</tbody>
</table>

In Table 6 we summarize the impact of an increasing reserve ratio on average firm size, average growth and variance of growth. To see more clearly how firms of different ages are influenced differently, we decompose the population of firms into three age groups, 1-20, 21-40, 41-60. Bank credit to firms decreases by 0.068% on average as the reserve ratio increases from zero to 20%. However, the average credit to firms below 20 years old (accounting for 81.13% of all firms) decreases by 0.138%, twice as much as the average decrease. At the same time, the average credit to older firms (age above 20, accounting for 18.87% of all firms) even increases by 0.22%. This implies that there is a reallocation of credit from young firms towards older ones. In other words, a decrease in aggregate credit depresses average firm size due to a tighter financial constraint on the young firms.

Average firm size first decreases and then increases. The increase comes from a strong

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The values of $V_0^E$ at $\mu = \{0, 0.2, 0.4\}$ are -8.3549, -8.3553 and -8.3554, respectively.
positive impact of an increasing average promised values on the credit availability as the reserve ratio becomes very large ($\mu = 0.4$). This effect is strong enough to overcome not only the propagation effect of a lower initial promised value, but also the negative equilibrium price effect on the optimal bank loan, $b(V^E)$. Moreover, this positive effect mainly influences the credit availability of older firms. This is confirmed by looking at the firm size of different age groups: Only for firms older than 20 years, banks’ credit expands, whereas for young firms credit shrinks as a result of the negative equilibrium price effect and the lower initial promised value. Furthermore, as the reserve ratio increases, firm growth decreases, and its variance decreases as well. This is because of a dominating equilibrium price effect: Since the efficient firm size is smaller as the reserve ratio rises, the range of bank loans decreases, and thus also the average growth and variance of growth.

In sum, except for unrealistic high levels of reserve ratios, firms operate in smaller size in an economy with higher a reserve ratio. In addition, there is a reallocation of credit from young firms towards older firms, implying a worse-off financial situation of the young firms. In the end, firms grow more slowly but more steadily as the reserve ratios increase. The lower growth rate of firms is accompanied by less volatile growth.

We verify numerically that it is indeed due to an increasing average promised values that drives up the average firm size. Specifically, we want to isolate the effect on average firm size due to the transition in promised values with the other two negative effects. To do this, we use the distribution of promised values calculated with transition function $V^E(\theta_s, V^E)$ under $\mu = 0$, in combination with the optimal bank loan $b(V^E)$ and the initial promised value $V^W(0)$ under $\mu = 0.4$, to get average bank loans. The value is smaller than the average bank loans under $\mu = 0$. This implies that the gap between this value and the actual average bank loan under $\mu = 0.4$ is due to a strong impact of increasing average promised values on the level of bank loans.
6 Concluding remarks

This paper investigates the impact of a decrease in aggregate credit supply – as a result of stricter reserve requirements – on firm dynamics, the credit availability at the firm level and the macroeconomic aggregates. We model a long-term lending relationship between banks and firms under asymmetric information. The resulting optimal dynamic contract – derived in recursive formulation – captures two salient features of financial markets: Firm-performance dependence of financial contracts, and long-term interactions between banks and firms to overcome asymmetric information. The financial market is endogenously incomplete due to informational asymmetries. This is reflected by tight financial constraints on young firms, which diminish gradually as firms get older. The model generates results that are consistent with empirical regularities: both the evolution of firm size distribution (in terms of promised values) and firm dynamics (i.e., firm size, growth rate of firms, and the volatility of growth) over a lifetime. We endogenize households’ occupational decision and determine the share of entrepreneurs in equilibrium. In the analysis of tightening aggregate credit supply, our paper highlights the non-trivial impact of households’ occupational choices on the macroeconomic aggregates and the credit availability of firms. We show that the negative impact of a higher interest rate on entrepreneurs’ lifetime expected utility is mitigated given the option of becoming a worker in the first place, whereas the impact on the aggregate output is amplified as the share of entrepreneurs decreases. Moreover, the model emphasizes a reallocation of credit from young firms towards old ones as a result of higher reserve ratios. There is a whole strand of literature that documents higher productivity among young firms, and how tighter financial constraints on these firms influence the macroeconomy. Following this argument, our model seems to imply that if the aggregate credit supply decreases, a compensatory lending policy to or a fiscal policy that aims at young firms is needed. However, to have a comprehensive analysis of the macroeconomic consequences, a possible extension is to introduce the age dependence of firm productivity or a persistent component in firm productivity realization (e.g., a Markov productivity shock). This is
computationally challenging, because the distribution of firms’ promised value within a cohort is no longer constant in a stationary equilibrium. Our paper takes a first step in this direction.
A Derivations

A.1 Actuarially fair pricing of annuities

The annuity deposits are priced actuarially fair, such that banks make zero profit from offering them to the workers. In other words, the aggregate amount of money received by the banks from workers plus the interest it generates within a period must be equal to what they give out in the next period. Formally, at time $t$,

$$
\sum_{\tau=0}^{\infty} (1 + r_{t+1})(1 - \Delta)\Delta^\tau p^A_{\tau,t+1} = \sum_{\tau=0}^{\infty} (1 - \Delta)\Delta^\tau+1 A_{\tau,t+1},
$$

where $(1 - \Delta)\Delta^\tau p^A_{\tau,t+1}$ is the aggregate payments of the workers of age $\tau$ at time $t$ to buy the annuity. This generates an interest with rate $1 + r_{t+1}$ in the next period. The aggregate amount is redistributed to all workers from last period who are still alive this period, which is $\Delta$-times the original size $(1 - \Delta)\Delta^\tau$ of each cohort. Therefore, zero-profit for banks implies that

A.2 Derivations of financial contract properties

The proofs for Proposition 1 and 2 and Lemma 1 and 2 follow the proofs on the optimal social insurance in Ljungqvist and Sargent (2000) which are based on Thomas and Worrall (1990).

A.2.1 Proof of Proposition 1

Proof. Using the definition in (12) and summing up $C_{s,s-1} + C_{s-1,s}$ we conclude: $C_{s,s-1} + C_{s-1,s} \geq 0$, which is equivalent to

$$
U(\theta_s R(b) - m_s, L^E) - U(\theta_s R(b) - m_{s-1}, L^E) \geq
U(\theta_{s-1} R(b) - m_s, L^E) - U(\theta_{s-1} R(b) - m_{s-1}, L^E)
$$

(A.1)
Since $\theta_s > \theta_{s-1}$ and given the strict concavity of the utility function in consumption, \eqref{A.1} is satisfied only if $m_s \geq m_{s-1}$. It then follows from $C_{s,s-1} \geq 0$ that $V^E_s \geq V^E_{s-1}$. 

\textbf{A.2.2 Proof of Lemma 1}

Proof. Without loss of generality, we prove from the local downward constraints $C_{s,s-1} \geq 0, \forall s \in S$, that for any $i > j, i, j \in S, C_{i,j} \geq 0$. The case of $i < j$ can be proved from the local upward constraints $C_{s,s+1} \geq 0, \forall s \in S$, using the same logic.

\textit{Proof with mathematical induction:}

For $n = 1$, $C_{j+n,j} \geq 0$ holds according to the local downward constraint. Suppose for $n \geq 1$, $C_{j+n,j} \geq 0, \forall j \in S$ holds; we need to prove that $C_{j+n+1,j} \geq 0$. For simplicity of notation denote $i = j + n$.

First, $C_{i,j} \geq 0$ and $C_{i+1,j} \geq 0$ are equivalent to the following inequalities:

\begin{equation}
U(\theta_i R(b) - m_i, L) + \beta \Delta V^E_i - U(\theta_i R(b) - m_j, L) - \beta \Delta V^E_j \geq 0, \\
U(\theta_{i+1} R(b) - m_{i+1}, L) + \beta \Delta V^E_{i+1} - U(\theta_{i+1} R(b) - m_i, L) - \beta \Delta V^E_i \geq 0.
\end{equation}

Summing up the two inequalities we have:

\begin{equation}
U(\theta_{i+1} R(b) - m_{i+1}, L) + \beta \Delta V^E_{i+1} - \beta \Delta V^E_j + \\
U(\theta_i R(b) - m_i, L) - U(\theta_{i+1} R(b) - m_j, L) - U(\theta_{i+1} R(b) - m_i, L) \geq 0. \tag{A.2}
\end{equation}

Using the strict concavity of the utility function, the fact $\theta_{i+1} > \theta_i$, and $m_i \geq m_j$ from Proposition 1, we have additionally the following inequality:

\begin{equation}
U(\theta_{i+1} R(b) - m_{i+1}, L) - U(\theta_{i+1} R(b) - m_j, L) \geq \\
U(\theta_i R(b) - m_i, L) - U(\theta_i R(b) - m_j, L). \tag{A.3}
\end{equation}

Adding (A.3) to (A.2) we have

\begin{equation}
U(\theta_{i+1} R(b) - m_{i+1}, L) + \beta \Delta V^E_{i+1} - \beta \Delta V^E_j - U(\theta_{i+1} R(b) - m_j, L) \geq 0.
\end{equation}

Namely, $C_{i+1,j} \geq 0$. 

\hfill \Box
A.2.3 Proof of Lemma 2

Proof. First, we prove by contradiction that the local downward constraints must bind. Suppose that there exists an optimal contract \( \{ b, m_s, V^{E}_s \} \in S \) such that for some \( i \in S \) the downward constraint does not bind (i.e., \( C_{i,i-1} > 0 \)). Then, the general procedure is as follows: We prove that there exists a mean-preserving contraction transformation on \( \{ V^{E}_j \}_{j=i,...,S} \) such that the new contract \( \{ b, m_s, \hat{V}^{E}_s \} \in S \), where \( \hat{V}^{E}_j = V^{E}_j \), for \( j = 1, 2, \ldots, i-1 \), fulfills all constraints. In particular, we make a transformation with \( \sum_{s \in S} \pi_s \hat{V}^{E}_s = \sum_{s \in S} \pi_s V^{E}_s \), and \( \hat{V}^{E}_j - \hat{V}^{E}_l \leq V^{E}_j - V^{E}_l \), \( \forall j,l \in S \), with at least one pair of \( \{ j,l \} \) giving strict inequality. In this case, under the assumption that \( P(V^{E}) \) is strictly concave, the banks’ profit increases strictly with the new contract. This contradicts the fact that \( \{ b, m_s, V^{E}_s \} \in S \) is an optimal contract.

Now we describe explicitly the procedure of performing a mean-preserving contraction transformation on the contract:

Keeping \( \{ m_{i-1}, m_i, V^{E}_{i-1} \} \) as before, we decrease \( V^{E}_i \) until \( C_{i,i-1} = 0 \). Since changing \( V^{E}_i \) will influence the local downward incentive constraints for \( s = i+1 \) and sequentially \( s = i+2, \ldots, S \), we decrease for each \( s = i+1, \ldots, S \), \( V^{E}_s \) such that \( C_{s,s-1} = 0 \). As a result we have a new sequence of future promised value \( \{ V^{E'}_s \} \in S = \{ V^{E}_{1}, V^{E}_{2}, \ldots, V^{E}_{i-1}, V^{E'}_{i}, V^{E'}_{i+1}, \ldots, V^{E'}_{S} \} \).

Now we add a positive constant, \( \bar{v} \), to the sequence of future promised value, such that the promise keeping constraint is regained. Let \( \hat{V}^{E}_s = V^{E'}_s + \bar{v} \). We have a new contract \( \{ b, m_s, \hat{V}^{E}_s \} \in S \).

First, note that the new contract fulfills the local upward constraints automatically given the strict concavity of the utility function and the fact that \( C_{s,s-1} = 0 \ \forall s \in S \) (see argumentation in the last part of this proof). In addition, the promise keeping constraint is still fulfilled due to the mean-preserving transformation, and the limited liability constraints are uninfluenced since \( b \) and \( \{ m_s \} \in S \) are unchanged. Finally, for any \( j = i, \ldots, S \), \( V^{E}_{j+1} \) must decrease at least as much as \( V^{E}_j \) to guarantee that \( C_{j+1,j} = 0 \). Therefore, for
any $j = i, \ldots, S$, $\bar{v} \leq V_j^E - V_{j'}^E$, indicating that $\hat{V}_j^E \leq V_j^E$ and remember that for $j = 1, 2, \ldots, i - 1 \hat{V}_j^E = V_j^E$. Since $\left\{ V_s^E \right\}_{s \in S}$ fulfills the credibility constraints, so does the new contract.

Further, notice from the procedure that the gap of the promised values between two successive states, $s$ and $s - 1$, is either unchanged or decreased, with a definite decrease in $\hat{V}_i^E - \hat{V}_{i-1}^E$. Following this we know $\forall j, l \in S, V_j^E - V_l^E$ is non-increasing. Thus, the new contract is a mean-preserving contraction. This contradicts that $\left\{ b, m_s, V_s^E \right\}_{s \in S}$ is an optimal contract. We know that the local downward constraints always bind.

Given that $C_{s,s-1} = 0, \forall s \in S$, rewriting the constraint we have

$$\beta \Delta (V_s^E - V_{s-1}^E) = U(\theta_s R(b) - m_{s-1}, L^E) - U(\theta_s R(b) - m_s, L^E)$$

Since $\theta_{s-1} < \theta_s, m_{s-1} \leq m_s$ and the utility function is strictly concave, we have

$$U(\theta_{s-1} R(b) - m_{s-1}, L^E) - U(\theta_{s-1} R(b) - m_s, L^E) \geq$$

$$U(\theta_s R(b) - m_{s-1}, L^E) - U(\theta_s R(b) - m_s, L^E) = \beta \Delta (V_s^E - V_{s-1}^E),$$

where strict inequality holds for $m_{s-1} < m_s$. Therefore, we have directly from this that the local upward constraint is never binding. Namely, $C_{s-1,s} > 0, \forall s \in S$.  

\[ \blacksquare \]

A.2.4 Proof of Proposition 2

Proof. The non-decreasing entrepreneurs’ utility is direct result of the binding local downward constraints.

The non-decreasing profit of banks is proved by contradiction. Suppose for the optimal contract, $\left\{ b, m_s, V_s^E \right\}_{s \in S}$, there exists $i, j \in S, i > j$, such that

$$-b + m_i + \frac{\Delta}{1 + r} P(V_i^E) < -b + m_j + \frac{\Delta}{1 + r} P(V_j^E).$$

Substituting $(m_i, V_i^E)$ with $(m_j, V_j^E)$ increases banks’ profit in state $i$. Since the downward constraint binds, $C_{i,j} = 0$, the terms of contracts, $(m_j, V_j^E)$, entitle the entrepreneurs
the same promised value as \((m_i, V^E_i)\). This means that we find an improvement that increases the profit of the banks without violating any constraints. This contradicts the optimality of the original contract. Therefore, in the optimal contract the banks’ profits cannot decline with a higher productivity realization.

\[\square\]

A.2.5 Proof of Proposition 3

We prove Proposition 3 using the F.O.C.s of the banks’ problem defined in A.4 For simplification, we derive the Lagrangian and the F.O.C. for the case of two states, \(S = \{l, h\}\) with \(\pi_h = \pi, \pi_l = 1 - \pi\). Substitute the repayments \(m_s\) with \(m_s = \theta_s R(b; r, w) - c_s\), the banks’ problem can be rewritten as the following:

\[
P(V^E; r, w) = \max_{b, (m_s, V^E_s), \pi_s \in S} -b + \sum_{s \in S} \pi_s \left[ \theta_s R(b; r, w) - c_s + \frac{\Delta}{1 + r} P(V^E_s; r', w') \right]
\]  

(A.4)

subject to

\[
V^E = \sum_{s \in S} \pi_s [U(c_s, L^E) + \beta \Delta V^E_s],
\]

(PK)

\[
U(c_i) + \beta \Delta V^E_i \geq U((\theta_i - \theta_j) R(b; r, w) + c_j, L^E) + \beta \Delta V^E_j, \ \forall i, j \in S,
\]

(IC)

\[
c_s \geq 0, \forall s \in S,
\]

(LL)

\[
V^E_s \in [V^E_{\min}, V^E_{\max}].
\]

(CC)

where in (IC), we substitute the current period consumption when misreporting using

\[
\theta_i R(b; r, w) - m_j = \theta_i R(b; r, w) - \theta_j R(b; r, w) - c_j = (\theta_i - \theta_j) R(b; r, w) + c_j.
\]

Let \(\lambda_1\) be the Lagrangian multiplier for (PK), \(\lambda_2\) and \(\lambda_3\) for (IC), \(\lambda_4\) and \(\lambda_5\) for (LL), \(\lambda_6, \lambda_7, \lambda_8, \lambda_9\) for (CC). The Lagrangian is given by:

50
\[ \mathcal{L} = \max_{ \{b, m_s, V_E^s\} \in S} -b + \sum_{s = \{l, h\}} \pi_s \left[ \theta_s R(b; r, w) - c_s + \frac{\Delta}{1 + r} P(V_s^E) \right] \\
+ \lambda_1 \left\{ \sum_{s = \{l, h\}} \pi_s [U(c_s, L^E) + \beta \Delta V_s^E] - V^E \right\} \\
+ \lambda_2 \left\{ U(c_h, L^E) + \beta \Delta V_h^E - U((\theta_h - \theta_l) R(b; r, w) + c_l, L^E) - \beta \Delta V_l^E \right\} \\
+ \lambda_3 \left\{ U(c_l, L^E) + \beta \Delta V_l^E - U((\theta_l - \theta_h) R(b; r, w) + c_h, L^E) - \beta \Delta V_h^E \right\} \\
+ \lambda_4 c_h + \lambda_5 c_l \\
+ \lambda_6 (V_{max}^E - V_l^E) + \lambda_7 (V_l^E - V_{min}^E) \\
+ \lambda_8 (V_{max}^E - V_h^E) + \lambda_9 (V_h^E - V_{min}^E), \]

Therefore, the F.O.C.s are

\[ \frac{\partial \mathcal{L}}{\partial c_h} = -\pi + (\lambda_1 \pi + \lambda_2) U'(c_h) - \lambda_3 U'(c_l h) + \lambda_4 = 0; \quad (A.5) \]

\[ \frac{\partial \mathcal{L}}{\partial c_l} = -(1 - \pi) + (\lambda_1 (1 - \pi) + \lambda_3) U'(c_l) - \lambda_2 U'(c_{hl}) + \lambda_5 = 0; \quad (A.6) \]

\[ \frac{\partial \mathcal{L}}{\partial V_h^E} = \frac{\pi}{1 + r} P'(V_h^E) + (\lambda_1 \pi + \lambda_2 - \lambda_3) \beta - \lambda_8 + \lambda_9 = 0; \quad (A.7) \]

\[ \frac{\partial \mathcal{L}}{\partial V_l^E} = \frac{1 - \pi}{1 + r} P'(V_l^E) + (\lambda_1 (1 - \pi) - \lambda_2 + \lambda_3) \beta - \lambda_6 + \lambda_7 = 0; \quad (A.8) \]

\[ \frac{\partial \mathcal{L}}{\partial b} = -1 + \mathbb{E}(\theta) R'(b; r, w) - [\lambda_2 U'(c_{hl}) - \lambda_3 U'(c_{hl})] (\theta_h - \theta_l) R'(b; r, w) = 0. \quad (A.9) \]

These F.O.C.s together with the complementary conditions characterize the conditions an optimal contract needs to fulfill. Now we prove Proposition \[3\] using these results and Lemma \[2\].

**Proof of Proposition \[3\]** Following from Lemma \[2\], the downward constraint always binds (\( \lambda_2 \geq 0 \)) whereas the upward constraint for \( m_s > m_{s-1} \) never does (\( \lambda_3 = 0 \)). Therefore, (A.9) can be rewritten as

\[ \mathbb{E}(\theta) R'(b; r, w) - \lambda_2 U'(c_{hl})(\theta_h - \theta_l) R'(b; r, w) = 1. \quad (A.10) \]
Compared with (15) which defines the efficient level of bank loans, \( b^* \), we have an additional term \( \lambda_2 U'(c_{hl})(\theta_h - \theta_l) R'(b; r, w) \), which comes from the (IC) constraint. It represents the marginal cost of maintaining entrepreneurs’ incentive of truth-telling. This cost is determined by two components: First, the gap of output in high and in low state. It represents part of the additional consumption entrepreneurs from misreporting. This gain increases as the level of bank loans increases. As a result, it make banks harder to maintain the constraint. Second, entrepreneurs’ current period consumption. When consumption is high, marginal utility from consumption is low. Thus, entrepreneurs are less sensitive to the additional consumption from misreporting, which means less incentive to misreport. Technically, as \( c_{hl} = \theta_h R(b) - m_l \) increases (e.g., if \( m_l \) decreases), \( U'(c_{hl}) \) decreases. Therefore, the marginal cost is lower. In the extreme case when \( c_{hl} \) approaches infinity (as is the case when \( V^E \) approaches \( V^E_{\max} \)), the marginal cost of maintaining incentives is almost zero. And the level of bank loans under asymmetric information is close to the efficient level. Otherwise, since \( \lambda_2 \geq 0 \) and \( U'(c) \geq 0 \), \( E(\theta) R'(b) \geq 1 \) and \( b \leq b^* \). This shows that due to the presence of asymmetric information, the level of bank loans is lower than the efficient level, and the firms are financially constrained.

\[ \Box \]

### A.3 Proof of the existence of a stationary firm distribution

In this appendix, we prove that there exists a stationary distribution of firms with respect to their today’s promised value, \( V^E \), \( \Psi(V^E) \).

Given the future promised value of firms (i.e., the policy function \( V''(V^E, \theta; r, w) \)) and the exogenous shock on survival and productivity, the transition function is given by \( T(V^E, B) : [V^E_{\min}, V^E_{\max}] \times B([V^E_{\min}, V^E_{\max}]) \to [0, 1] \), where \( T(V^E, B) \) indicates the probability of a firm with today’s promised value \( V \) to get a promised value \( V' \in B \) for tomorrow. Formally, we write the transition function as

\[
T(V^E, B) = \Delta \sum_{s \in S} \pi_s \mathbb{1}(V'(V^E, \theta_s; r, w) \in B) + (1 - \Delta) \mathbb{1}(V_0^E(r, w) \in B).
\]
We need to prove that the transition function \( T(V^E, B) \) is monotone, satisfies the Feller property and the mixing condition. Suppose \( L \) is the operator associated with \( T(V^E, B) \). Given a function \( f(x) \) (measurable on \([V_{min}^E, V_{max}^E], B([V_{min}^E, V_{max}^E])\)), either bounded increasing or bounded continuous,
\[
(Lf)(V^E) = \int f(x)T^E(V^E, dx) = \Delta \sum_{s \in S} \pi_s f(V'(V^E, \theta_s; r, w)) + (1 - \Delta) f(V^E_0(r, w)).
\]
Since the policy function \( V'(V^E, \theta; r, w) \) is increasing and continuous in \( V^E \), \( (Lf)(V^E) \) is increasing (continuous), if \( f(x) \) is increasing (continuous).

Take \( V^* = V^E_0(r, w), N = 1, \) and \( \epsilon = 1 - \Delta. \) Similarly to the argument for households, we have \( T(V^E, B) \) satisfies the mixing condition.

Given the idiosyncratic shock on firm productivity each period and the exponential increment of the history of such shocks, an explicit characterization goes very messy very fast. Therefore, we simply denote the stationary distribution as \( \Psi(V^E) \).

### A.4 Derivation of the good market clearing condition

The goods market is cleared if aggregate output equals the sum of consumption of all households and aggregates investment (i.e., replacement of depreciated capital in stationary case). Remember from (30) that the formal condition is \( \lambda Y = \lambda C^E + (1 - \lambda) C^W + \lambda \delta K^D. \) To prove that the goods market clearing condition can be derived from the other equations, we need only to prove that the RHS of (30) can be simplified to \( \lambda Y. \)

We aggregate the consumption of entrepreneurs \( C^E \) and workers \( C^W \). Plugging in entrepreneur’s consumption \( c(.) \) from (7) into (24) (using \( R(b) = R(b(V^W)) \)) and \( m_s = m(V^E, \theta_s) \) we obtain for aggregate consumption of entrepreneurs:
\[
C^E = \sum_{\tau=0}^{\infty} (1 - \Delta)\Delta^\tau \sum_{s \in S} \pi_s \int \left[ \theta_s R(b(V^E)) - m(V^E, \theta_s) \right] d\Psi_\tau(V^E) = Y - M, \quad (A.11)
\]
where the second equality follows from (23) and (22).
A cohort $\tau$ worker’s consumption is $c_\tau = w l_\tau + A_\tau - p^A A_{\tau+1}$ (follows from the budget constraint (3)). Aggregation gives

$$C^W = \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau (w l_\tau + A_\tau - p^A A_{\tau+1})$$

$$= w L^S - D + \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^\tau A_\tau$$

$$= w L^S - D + \Delta \sum_{\tau=0}^{\infty} (1 - \Delta) \Delta^{-1} A_\tau$$

$$= w L^S - D + \Delta \frac{1}{p^A} D$$

$$= w L^S + r D,$$  \hspace{1em} (A.12)

where the second and forth equality follow from (18) and (17) and the annuity price $p^A = \frac{\Delta}{1+r}$ from (1) is used.

Entrepreneurs have the bank loans to finance the production costs. All money is used for production costs. Thus, the constraint in entrepreneurs’ decision problem in (5) is binding so that we have $b(VE) = w l^*(VE) + (r + \delta) k^*(VE)$. In the aggregate, this means

$$B = w L^D + (r + \delta) K^D.$$  \hspace{1em} (A.13)

Plugging (A.11) and (A.12) into the RHS of (30) gives:

$$\text{RHS} = \lambda (Y - M) + (1 - \lambda) (w L^S + r D) + \lambda \delta K^D$$

$$= \lambda Y - \lambda M + (1 - \lambda) w L^S + (1 - \lambda) r D + \lambda r E - \lambda \delta K^D$$

$$= \lambda Y + \lambda w L^D + \lambda (r + \delta) K^D - \lambda (M + r E)$$

$$= \lambda Y + \lambda B - \lambda B$$

$$= \lambda Y = \text{LHS},$$

where the equilibrium conditions (28) and (29) were used in the third equality and (25) and (A.13) were used in the third. This closes the proof that the goods market clearing condition can be derived from clearing in the capital and labor markets.
B Optimization problem: Workers and entrepreneurs

B.1 Workers’ optimal decisions

Figure 7 depicts, as a function of the current period deposit wealth $A$, the workers’ optimal consumption $c(A)$, the labor supply $l(A)$ and the saving decision $A'(A)$ corresponding to the policy functions given in (4) and the lifetime expected utility $V^W(A)$ for given $w$ and $r$.

![Graphs showing consumption, labor supply, savings, and value function as functions of deposit wealth A.](image)

Figure 7: Solution to worker’s problem

Because of income effect, workers with higher deposits consume more today, save more

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38See Appendix C.1 for the procedure to solve the recursive workers’ problem numerically.
for tomorrow, and supply less labor. Their lifetime expected utility, captured by the 
value function $V^W(A)$, is an increasing function in $A$, indicating that workers are better 
off if endowed with more wealth $A$. In addition, the envelop theorem implies that $A' > A$ 
if $\beta(1 + r) > 1$, which is the case in our numerical analysis. In other words, workers 
postpone their consumption and increase saving overtime with a small future discount 
rate. Nevertheless, this does not mean that the aggregate deposit in the economy will 
go to infinity in equilibrium. Remember that in a stationary equilibrium, there is a 
constantly inflow of newborn workers and exit of the old ones, with the aggregate deposit 
defined in (17). As long as the growth rate of deposit is lower than the exit rate of the 
workers – as is the case in the quantitative analysis, the aggregate deposit is finite.

B.2 Entrepreneurs’ optimal capital and labor employment

For a given level of bank loans $b$ and factor prices $w$ and $r$, the entrepreneur’s optimal 
capital input and labor employment based on the decision problem in (5) are linear in $b$ 
(see plot in Figure 8):

$$k^* = \frac{1}{r + \delta} \frac{\alpha_k}{\alpha_k + \alpha_l} b \quad \text{and} \quad l^* = \frac{1}{w} \frac{\alpha_l}{\alpha_k + \alpha_l} b.$$  

(B.1)

The corresponding firm output is thus

$$R(b) = \left( \frac{\alpha_k}{r + \delta} \right)^{\alpha_k} \left( \frac{\alpha_l}{w} \right)^{\alpha_l} \left( \frac{b}{\alpha_k + \alpha_l} \right)^{\alpha_k + \alpha_l} \bar{a}.$$  

(B.2)

Following from equation (15), the efficient level of bank loans is thus given by

$$b^* = (\alpha_k + \alpha_l) \left[ \bar{a} \left( \frac{\alpha_k}{r + \delta} \right)^{\alpha_k} \left( \frac{\alpha_l}{w} \right)^{\alpha_l} \right]^{\frac{1}{r - \alpha_k - \alpha_l}}.$$  

(B.3)

39Technically, this is the sufficient condition for the convergence of the infinite sequence, 
$\{(1 - \Delta)\Delta^\tau p^A g(A, r, w)\}_{\tau=0}^\infty$. 

56
In this section we describe the dynamic programming algorithm for solving the partial equilibrium (i.e., workers’ decision problem in Section C.1 and banks’ optimal contract in Section C.2), the procedure to simulate the entrepreneurs’ life path and calculate the aggregate variables (Section C.3), and the algorithm to calculate the unique stationary general equilibrium (Section C.4). All computations are done with Matlab.

C.1 Workers

1. Use constant \( r \) and \( w \).

2. Set a grid for the state variable \( A \). \( A_{\text{grid}} \) denotes the grid points of \( A \). We set \( A = [0, 10] \) and generate \( nA = 50 \) Chebyshev grid points on the interval. We manually replace the lowest Chebyshev point with the lower bound of \( A = 0 \).

3. Give an initial guess for the functional form of the value function, \( V_{W}^W(A; r, w)^0 \), of the policy functions \( l(A)^0 \) and \( A'(A)^0 \) and of \( c(A)^0 \). We use \( V_{W}^W(A_i)^0 = \)

---

Even though the functions (e.g., value functions, \( V_{W}^W(A) \) and \( P(V^E) \)) are continuous per se, they...
\[-\exp(-A_i) - 0.1, c(A_i)^0 = 0.1, I(A_i)^0 = 0.6 \text{ and } A'(A_i)^0 = 0 \text{ for each } A_i \in Agrid, i \in \{1, \ldots, nA\}.\]

4. Solve on each grid point \(A_i \in Agrid, i \in \{1, \ldots, nA\}\), the worker’s problem in (2) subject to (3), \(c(A_i) \geq 0, l(A_i) \in [0, 1] \text{ and } A'(A_i) \geq 0\). This gives us the optimal solution of the system, \(\{c(A_i)^1, l(A_i)^1, A'(A_i)^1\}\) and the corresponding updated value function \(V^W(A_i; r, w)^1\) at \(A_i \in Agrid, i \in \{1, \ldots, nA\}\). To calculate the updated value function, we interpolate on \(V^W(A; r, w)^0\) to get values for \(A'(A)\) which lie between two \(Agrid\)-points.\(^{41}\)

5. Compare the two successive iterations of value functions, \(V^W(A; r, w)^1\) with \(V^W(A; r, w)^0\), by defining a distance measure \(d_{V^W}\), such that

\[
d_{V^W} \equiv \max_{i \in \{1, \ldots, nA\}} |V^W(A_i; r, w)^1 - V^W(A_i; r, w)^0|.
\]

If \(d_{V^W} \leq \epsilon_P\), the optimal solution from the current iteration solves the workers’ problem and go to Step 5.\(^{43}\) If \(d_{V^W} > \epsilon_P\) go to Step 3 by updating \(V^W(A; r, w)^0 = V^W(A; r, w)^1, c(A)^0 = c(A)^1, l(A)^0 = l(A)^1 \text{ and } A'(A)^0 = A'(A)^1\) as the new starting values.

6. Save the value function \(V^W(A_i; r, w) = V^W(A_i; r, w)^1\), the \(A'(A_i) = A'(A_i)^1\) and \(l(A_i) = l(A_i)^1\) and \(c(A_i) = c(A_i)^1\) for each \(A_i \in Agrid, i \in \{1, \ldots, nA\}\).

C.2 Financial contract

1. Use constant \(r\) and \(w\). 

\(^{41}\) We apply the \textit{fmincon}-command, which finds the minimum of a constrained nonlinear multivariable function using the interior point algorithm.

\(^{42}\) We use spline interpolation, which is a cubic interpolation of the values of neighbor-points.

\(^{43}\) We set the tolerated distance for ending the iterations, \(\epsilon_P = 0.0001\).
2. Set a grid for the state variable $V^E$ on the interval $[V^E_{min}, V^E_{max}]$. $V^E_{grid}$ denotes the $nV^E = 50$ Chebyshev grid points of $V^E$ on the interval. We manually replace the lowest Chebyshev point with the lower bound $V^E_{min}$.

3. Make an initial guess of the functional form of the value function, $P(V^E; r, w)^0$, and of the policy functions, $\left\{ b(V^E), m_s(V^E), V^E_s(V^E)^0 \right\}_{s \in \{h,l\}}$. We use $P(V^E; r, w)^0 = \log(-V^E_i)$, $b(V^E_i)^0 = 1$, $m_h(V^E_i)^0 = 3$, $m_l(V^E_i)^0 = 1$, $V^E_h(V^E_i)^0 = V^E_i(V^E_i)^0 = V^E_i$ for each $V^E_i \in V^E_{grid}$, $i \in \{1, ..., nV^E\}$.

4. Solve for each $V^E_i \in V^E_{grid}$, $i \in \{1, ..., nV^E\}$ the optimal contract in $[A.4]$ subject to (PK), (IC), (LL) and (CC). This gives the optimal contract at each $V^E_{i} \in V^E_{grid}$, $i \in \{1, ..., nV^E\}$, $\left\{ b(V^E_i)^1, m_s(V^E_i)^1, V^E_s(V^E_i)^1 \right\}_{s \in \{h,l\}}$, and thus the updated value function, $P(V^E; r, w)^1$.

5. Compare the two successive iterations of value functions, $P(V^E; r, w)^1$ with $P(V^E; r, w)^0$, by defining a distance measure $d_P$, such that

$$d_P = \max_{i \in \{1, ..., nV^E\}} \left| P(V^E_i; r, w)^1 - P(V^E_i; r, w)^0 \right|.$$

If $d_P \leq \epsilon_P$, then take the current iteration of value function and policy functions as the solution and go to Step 6. If $d_P \gt \epsilon_P$, start over with Step 3 by updating $P(V^E; r, w)^0 = P(V^E; r, w)^1$ and $b(V^E) = b(V^E)_1$, $m_h(V^E) = m_h(V^E)_1$, $m_l(V^E) = m_l(V^E)_1$, $V^E_h(V^E)^0 = V^E_h(V^E)^1$ and $V^E_l(V^E)^0 = V^E_l(V^E)^1$ as the new starting value for the next iteration.

6. Save the value function $P(V^E; r, w) = P(V^E; r, w)^1$ and the optimal contract $b(V^E) = b(V^E)^1$, $m_h(V^E) = m_h(V^E)^1$, $m_l(V^E) = m_l(V^E)^1$, $V^E_h(V^E) = V^E_h(V^E)^1$ and $V^E_l(V^E) = V^E_l(V^E)^1$.

---

44 For, (IC) we put in the constraint only the binding local downward constraint, since by the result of Lemma 2 the local upward constraint is never binding for the optimal contract.

45 As in the algorithm for solving the workers’ problem, we apply the fmincon-command to solve for the optimal contract at each grid point and use spline interpolation to calculate the value function for the next iteration.
C.3  Life path simulation and equilibrium variables

In this appendix, we simulate entrepreneurs’ life paths and calculate the aggregate variables related to entrepreneurs by combining the optimal solution from Section C.2 and the entrepreneurs’ decision. Moreover, we calculate workers’ aggregate deposits $D$ and the labor supply $L^S$ according to the straightforward analytical expressions (17) and (18). Using the aggregate variables, we calculate equilibrium values for the entrepreneurial share.

1. Simulate for $N^E = 10,000,000$ entrepreneurs’ age and life paths with history of productivity realizations. We use two random numbers, $u_i^t$ and $o_i^t$, to denote entrepreneur $i$’s productivity realization and death/survival at time $t$, respectively. The procedure is as follows:

(a) Start from entrepreneur $i = 1$, period $t = 1$.

(b) Use a random number generator to generate two numbers $o_i^1$ and $u_i^1$, which are uniformly distributed on the interval $[0, 1]$.

(c) If $u_i^1 < \pi_l$, save the entrepreneur’s productivity realization in this period as low, $\theta_i^t = \theta_l$, and otherwise as high $\theta_i^t = \theta_h$.

(d) If $o_i^t < \Delta$, the entrepreneur survives to the next period, increase $t$ by one and

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46 The numerical procedure described in this section is performed for given interest rate and wage rate, $\{r, w\}$.

47 Calculating the aggregate deposits and the aggregate labor supply according to analytical expressions instead of applying life path simulation is simply to save computational time. Notice that to guarantee that the equilibrium factor prices approximate the true values under acceptable computational error, $\varepsilon_{GE}$, we only need to make sure that the computed aggregate values approximate the true values at higher precision, irrespective of the way they are calculated. And this is guaranteed in Step 4 for the aggregate deposits and the aggregate labor supply (i.e., $\varepsilon_L = 0.0001 < \varepsilon_{GE}$). For the aggregate capital and labor demand, on the other hand, we have checked that by increasing $N^E$ to ten times of the current number, the changes of these two aggregate values are below $\varepsilon_L = 0.0001$ as well.

48 We set $\pi_l = 1/2$ and $\Delta = 0.92$. 

60
go back to Step 1b. If \( \phi_i^t > \Delta \), the life path stops. Save her year of life, \( A^i = t \), go to Step 1b and simulate for the next entrepreneur \( i + 1 \).

(c) Save all entrepreneurs’ years of life, \( \{A^i\}_{i=1}^{N^E} \), and the sequence of productivity realizations, \( \{\theta^{A^i}_i\}_{t=1}^{i}, i = 1, \ldots, N^E \).

2. Using the simulation of the life paths from Step 1 and the policy function from Section C.2, we determine the corresponding promised value \( V^i \), bank loans \( b(V^i) \) and the repayments \( m(V^i, \theta^{A^i}_i) \) for each entrepreneur \( i \) in their last period in life \( t = A^i \). Notice that the promised utility relevant for calculating the bank loans and repayments is the value at the beginning of the last period. This means that the productivity realization in \( t = A^i, \theta^{A^i}_i \), is only used for calculating the repayments (see Step 2d). Specifically, the procedure is as follows:

(a) Set \( V_0^i = V^W(0; r, w) \) using the workers’ value function from Section C.1.

(b) Start from entrepreneur \( i \), period \( t = 1 \).

(c) If \( t \leq A^i - 1 \), the promised utility of entrepreneur \( i \) at the beginning of period \( t \) is \( V_t^i = V^E_s(V_{t-1}^i) \), where \( V^E_s(V) \) is entrepreneurs’ transition function solved in Section C.2. Repeat the step until the condition is no longer satisfied.

(d) Calculate the optimal banks loans and repayments, \( \{b^i, m^i\}_{i=1}^{N^E} \), using the policy functions solved in Section C.2, the productivity realization in the last period of life, \( \{\theta^{A^i}_i\}_{i=1}^{N^E} \) from Step 1, and the promised utility from Step 2c, \( \{V_{A^i-1}^i\}_{i=1}^{N^E} \). Specifically, \( b^i = b(V^i_{A^i-1}) \) and \( m^i = (V^i_{A^i-1}, \theta^{A^i}_i) \).

3. Calculate the aggregate bank loans \( B \) and repayments \( M \). Specifically, we aggregate banks loans \( b^i \) and repayments \( m^i \) over all entrepreneurs \( i = 1, \ldots, N^E \) and divide the two sums by \( N^E \) to normalize the mass of the population to 1. Further, we calculate the aggregate capital demand \( K^D \) and aggregate labor demand \( L^D \) according to \( (B.1) \).

4. Calculate workers’ aggregate deposits \( D \) and the labor supply \( L^S \) according to equation \( (17) \), \( (18) \) and the optimal decisions solved in C.1. Specifically, start from
the deposits and labor supply of workers of age \( t = 1 \), \( \text{Sum}_A = p_A A'(A_{t-1}) \) and \( \text{Sum}_L = l(A_{t-1}) \), weighted by their population size, \( (1 - \Delta)\Delta^{t-1} \). Constantly add to \( \text{Sum}_A \) and \( \text{Sum}_L \) the weighted deposits and labor supply of the older generation, until the differences between the sums in two successive iterations is below \( \epsilon_L = 0.0001 \), respectively.

5. Determine the share of entrepreneurs from the labor market condition, \( \lambda = \frac{L_D}{L_D + L_S} \), the zero-profit condition, \( P(V^w(0; r, w); r, w) \), and the excess capital demand, \( X(r, w) \equiv \lambda K^D - (1 - \lambda)D - \lambda \frac{B - M}{r} \). Notice that all endogenous variables, \( \{\lambda, K^D, D, B, M\} \) are functions of the factor prices, \( (r, w) \).

### C.4 Numerical procedure for general equilibrium

In this section, we characterize the procedure for solving the general equilibrium (i.e., the factor prices \( (r, w) \)). The theoretical ground and intuition of the algorithm are given in section D. We will mention banks’ profit at \( V^E = V^W(0; r, w) \), \( P(V^W(0; r, w); r, w) \), repeatedly in this section. To save notation we write \( \Pi(r, w) \) (or \( \Pi \) when the specific values of \( (r, w) \) are irrelevant) instead of the full expression.

1. Start with \( (r_0, w_0) \) as an initial guess for the equilibrium factor prices. Calculate the partial derivatives of the banks’ profit, \( \Pi \), and of the excess capital demand, \( X \), at \( (r', w') \). Denote the partial derivatives as \( S^\Pi_r, S^\Pi_w, S^X_r, \) and \( S^X_w \), respectively.

We set \( \epsilon_L << \epsilon_{GE} \) so that the equilibrium is not susceptible to calculation error in the workers’ aggregate variables.

To approximate the partial derivatives, we calculate the value of \( \Pi(r, w) \) and \( X(r, w) \) at \( (r', w') \), \( (r' + \varepsilon, w') \) and \( (r', w' + \varepsilon) \), respectively. Applying the definition of partial derivatives, we have \( S^\Pi_r \approx \frac{\Pi(r' + \varepsilon, w') - \Pi(r', w')}{\varepsilon}, S^\Pi_w \approx \frac{\Pi(r', w' + \varepsilon) - \Pi(r', w')}{\varepsilon}, S^X_r \approx \frac{X(r', w' + \varepsilon) - X(r', w')}{\varepsilon}, \) and \( S^X_w \approx \frac{X(r', w' + \varepsilon) - X(r', w')}{\varepsilon} \). Notice that this procedure is time consuming, because we need to calculate through the entire model at each combination of \( (r, w) \). Since the searching region for the general equilibrium is relatively small (within interval of magnitude 0.01), the change in partial derivatives is small. Therefore, we use these values as an approximation of the partial derivatives in all iterations to save computational time.
2. Set \((r', w') = (r_0, w_0)\).

3. Calculate \(V^W(0; r', w')\) according to Algorithm C.1 and \(P(V^E; r', w')\) according to algorithm C.2.

4. If \(|\Pi(r', w')| < \epsilon_{GE}\), the banks’ profit is close enough to zero and go to Step 6. Otherwise, denote the interest rate such that \(\Pi(r', w') = 0\) as \(\hat{r}'\). If \(\Pi(r', w') > 0\), the lower bound of the interval where \(\hat{r}'\) must lie is \(r'\). Save \(r_L = r'\) and \(\Pi_L = \Pi(r', w')\). If \(\Pi(r', w') > 0\) the upper bound of the interval is \(r'\). Save \(r_U = r'\) and \(\Pi_U = \Pi(r', w')\).

5. If both the upper and the lower bound are found, let \(r_0 \equiv r_U \Pi_L - r_L \Pi_U / \Pi_L - \Pi_U\). Otherwise, \(r_0 \equiv \Pi(r', w') / S^P\). Go to Step 2.

6. Calculate the excess capital demand \(X(r', w')\) and share of entrepreneurs \(\lambda(r', w')\) according to Section C.3.

7. If \(|X(r', w') - \Pi(r', w')| < \epsilon_{GE}\), then the \((r', w')\) and the corresponding \(\lambda(r', w')\) in the current iteration are the equilibrium. Otherwise, if \(X(r', w') > 0\) the lower bound of the equilibrium wage is \(w'\). Save \(w_L = w'\) and \(X_L = X(r', w')\). If \(X(r', w') > 0\) the upper bound of the equilibrium wage is \(w'\). Save \(w_U = w'\) and \(X_U = X(r', w')\).

8. If both the upper and the lower bound are found, let \(w_0 \equiv (w_L + w_U) / 2\), and.

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51 We set \(\epsilon_{GE} = 0.001\).

52 This is only symbolic for the sake of describing the algorithm.

53 Notice that by setting the criterion as \(|X(r, w) - \Pi(r, w)| < \epsilon_{GE}\) instead of \(|X(r, w)| < \epsilon_{GE}\), we decreases the computational error of the equilibrium factor prices. Essentially, we want to avoid the case when both \(X(r, w)\) and \(\Pi(r, w)\) are marginally below \(\epsilon_{GE}\) but of the opposite sign. By analysis similar as illustrated in Figure 9 this deviates the numerical solution from the true values much more than if both \(X(r, w)\) and \(\Pi(r, w)\) are marginally below \(\epsilon_{GE}\) but of the same sign.

54 An explanation of why the two cases correspond to the upper and the lower bound is given in footnote and the corresponding part in the main text.
\[ r_0 \equiv r' - \left( \Pi(r', w') + S_w^P(w_0 - w') \right) / S_r^P. \] Otherwise, we set

\[
\begin{pmatrix} r_0 \\ w_0 \end{pmatrix} \equiv \begin{pmatrix} r' \\ w' \end{pmatrix} - A^{-1}b, \tag{C.1}
\]

where \( A = \begin{pmatrix} S_r^P & S_w^P \\ S_r^X & S_w^X \end{pmatrix} \), and \( b = \begin{pmatrix} \Pi(r', w') \\ X(r', w') \end{pmatrix} \).\(^{55}\) Go to Step 2.

\[ D \quad \text{Theoretical ground and intuition of Algorithm C.4} \]

The theoretical ground of the stationary general equilibrium searching in Algorithm C.4 is based on the continuity of the aggregate variables and the values functions with respect to \((r, w)\), and numerical properties of the zero-profit condition, \( \Pi \equiv P(V^W(0; r, w); r, w) \), and the excess capital demand, \( X \equiv \lambda(r, w)K^D(r, w) - (1 - \lambda(r, w))D(r, w) - \lambda(r, w)E(r, w) \).\(^{56}\)

**Property 1.** The zero-profit condition and the excess capital demand are both decreasing in \( r \) and \( w \) (at least locally around the equilibrium values).

This means that the partial derivatives of \( \Pi(r, w) \) and \( X(r, w) \) with respect to \( r \) and \( w \) are negative:

\[ \Pi_r < 0, \quad \Pi_w < 0, \quad X_r < 0 \text{ and } X_w < 0. \tag{D.1} \]

In a \((w, r)\)-diagram, the slope of the iso-profit curve and of the iso-excess demand curve are given respectively by

\[ S_{\Pi} = -\frac{\Pi_w}{\Pi_r} \quad \text{and} \quad S_X = -\frac{X_w}{X_r}. \tag{D.2} \]

Therefore, equation (D.1) implies that both loci are downward sloping (i.e., \( S_{\Pi} < 0 \) and \( S_X < 0 \)). In addition, a northeast shift of the locus (i.e., an increase in \( r \) and \( w \)) decreases the corresponding value of the respective iso-curve.

\(^{55}\) We apply Taylor’s expansion on \( \Pi(r, w) \) and \( X(r, w) \). Namely, \( \Pi(r_0, w_0) \approx \Pi(r', w') + \frac{\partial \Pi}{\partial r}(r_0 - r') + \frac{\partial \Pi}{\partial w}(w_0 - w') \), and \( X(r_0, w_0) \approx X(r', w') + \frac{\partial X}{\partial r}(r_0 - r') + \frac{\partial X}{\partial w}(w_0 - w') \). Setting \( \Pi(r_0, w_0), \Pi(r', w'), \) and \( X(r_0, w_0) \) to be zero we get equation (C.1).

\(^{56}\) We thank Josef Falkinger for pointing out to us the basis of this section.
Furthermore, the relative position of the two loci is determined by the following property.

**Property 2.** The gap between the two equilibrium conditions, \( G \equiv X - \Pi \), is decreasing in \( r \) and increasing in \( w \).

Property 2 implies that the iso-profit curve is steeper than the iso-excess demand curve at all combination of \((r, w)\) locally. To see this, note that the slopes of the iso-profit and iso-excess demand curves are given by equation (D.2). Since Property 2 indicates that the partial derivatives satisfy \( G_r < 0 \) and \( G_w > 0 \), we have

\[
\Pi_r > X_r \quad \text{and} \quad X_w > \Pi_w. \tag{D.3}
\]

Therefore, the slopes of the two loci satisfy \(|S_X| < |S_\Pi|\). A direct implication is the single-crossing property of the two loci. If the two curves ever cross they cross only once. This establishes the uniqueness of the stationary equilibrium. Furthermore, the properties of the iso-curves indicate the direction for approaching the equilibrium from any off-equilibrium point.

Figure 9 illustrates of iso-profit and iso-excess demand curves and gives an intuition of the algorithm to find the equilibrium, Eq.. Suppose that at an initial guess \((r_0, w_0)\) (e.g., point \(A\)) the value of the iso-profit is \(\Pi(r_0, w_0) > 0\). First, we approach the \(\Pi = 0\) locus by changing \(r\) to \(r^*_0\), s.t. \((r^*_0, w_0)\) is on the locus (Step 4 and 5 in Algorithm C.4). Then

\[57\] The fact that the two curves cross (i.e., the existence of the equilibrium) is guaranteed in the numerical practice. In the region we search for the equilibrium, there always exist combinations of \((r,w)\) on the zero-profit locus, s.t. \(X(r, w) > 0\), and combinations, s.t. \(X(r, w) < 0\). Since the zero-excess-capital-demand locus must lie between the loci that pass through the above mentioned two types of combinations, the zero-profit locus and the zero-excess-capital-demand locus cross.

\[58\] Since zero is unachievable numerically, we use \(|\Pi(r_0^*, w_0)| < \epsilon_P\) as a criterion for approximation. This applies for the excess capital demand \(X(r, w)\) as well. In addition, due to the unknown functional form of \(\Pi\), it is impossible to calculate the exact increase in \(r_0\) ex ante (i.e., \(r^*_0 - r_0\)). This means that there may be back and forth in the adjustment of \(r\). To guarantee that the target \(r_0^*\) is found in finite iterations, we record the upper and the lower bound of region where \(r_0^*\) lies in each iteration, and use binary search as is described in Step 5 and 8 in Algorithm C.4.
at \((r^*_0, w_0)\) the excess capital demand \(X(r^*_0, w_0)\) can be positive, negative or 0. In the last case we have found the equilibrium \(Eq.\) directly. Now suppose \(X(r^*_0, w_0) < 0\) (e.g., point \(C\)). Equation \((D.1)\) and \((D.3)\) suggest that the stationary equilibrium lies south-east of \(C\). Therefore, we shift \((r, w)\) - \(r \downarrow, w \uparrow\) - along the locus of the iso-profit curve \(\Pi = 0\) until the excess demand increases to 0 \((C \to Eq.)\).

\[59\] This also means that \(w_0 < w_{Eq}\). Therefore, \(w_0\) is one lower bound of the equilibrium wage. We will update the lower bound if a new \(w'\), s.t. \(w_0 < w' < w_{Eq}\) is found. The arguments apply for the upper bound as well.

\[60\] Similar to the situation described in footnote \([58]\) it is not possible to find the correct adjustment in \((r, w)\) in one step. Several iterations may be needed and we apply similar technique (i.e., recording upper and lower bounds and updating \(w\) in each iteration with binary search) to guarantee that the equilibrium is found in finite iterations. In addition, as we change \((r, w)\) in each iteration, we need to make sure that the change is along the locus of the iso-profit curve \(\Pi = 0\). Otherwise, we need to apply the first step again.
Figure 9: Iso-profit and iso-excess demand curves

Notes: Note that we do not know the curvature of the two curves. Below the two solid lines profit and excess demand are positive and above they are negative.

E Intuition for convergence to stationary equity level

From the characteristics of the optimal contract (Figure 2), we notice that at low levels of promised values $V^E$ expected repayments, $\pi l m_t(V^E) + (1-\pi) m_h(V^E)$, from entrepreneurs to banks exceed the level of bank loans $b(V^E)$ and that the opposite holds at high levels of promised values (see Figure 10). Intuitively, this means that banks receive a positive net cash flow from entrepreneurs with low promised values.

This positive net flow accrues to banks’ equity. This is supplied as capital on the capital market and generates returns, which lead to a further accumulation of equity. In contrast, banks expect a negative net cash flow from firms with high promised values, which
detracts banks’ equity.

Figure 10: Bank loans, $b(V^E)$, and expected repayments, $\pi_m l(V^E) + (1 - \pi_l) m_h(V^E)$

With this in mind, we can now intuitively describe the process of development of banks’ equity level from the very beginning of time with no population to the stationary equity level $E$. Suppose the banks are endowed with $E_0$ at the beginning of time when there is no population in the economy, yet. As population starts, there is a new-born cohort of entrepreneurs (and workers) with promised values $V^W(0; r, w) = V_0^E$. Entrepreneurs sign contracts with banks, which entitle them to banks loans and which ask for repayments. At the beginning of their lives, when entrepreneurs are at low levels of promised values they must give positive net cash flows to banks. Hence, banks start accumulating equity. With age, the average promised value of entrepreneurs increases (see firm dynamics in Figure 4 and 5) and reaches eventually levels where banks loans are larger than expected repayments. This reduces banks’ equity. In addition, as the economy evolves, there are more overlapping cohorts – with younger cohort making positive and older cohorts making negative net cash flows to banks. In aggregation there is an accumulation of total bank’s equity. Finally, in the stationary equilibrium the accumulation of banks’ equity come to a halt so that the equity level stays constant. This means, in equilibrium negative aggregate net payments from entrepreneurs are exactly covered by the interest generated on banks’ equity.

Assume for simplicity that during the process of development interest rate and wage are fixed at some level (e.g., the equilibrium level $(r, w)$).
F Figures

F.1 Illustration of productivity shock

Figure 11: Life path I

Figure 12: Life path II
F.2 Development of entrepreneurs’ bank loans and repayment

Figure 14: Development of entrepreneurs’ bank loans
Figure 15: Development of entrepreneurs’ repayments to banks

Notes: Most of the subplots exhibit two distinct levels of repayments. This reflects the fact that within one cohorts firms may have high or low productivity realizations.
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